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Data-driven system modeling and optimal control for nonlinear dynamical systems

by

Apurba Kumar Das

A thesis submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Electrical Engineering (Systems and Controls)

Program of Study Committee:
Umesh Vaidya, Major Professor
Venkataramana Ajjarapu
Farzad Sabzikar

The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this thesis. The Graduate College will ensure this thesis is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University

Ames, Iowa

2019

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DEDICATION

I would like to dedicate this thesis to three important women of my life, my mother Kabita Rani Deb, my wife Sudipta Saha Suma and my newborn daughter Amelia Samadrita Das.

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ABSTRACT

With the increasing complexity of modern industry processes, robotics, transportation, aerospace, power grids, an exact model of the physical systems are extremely hard to obtain whereas abundant of time-series data can be captured from these systems. This makes it a important and demanding research area to investigate feasibility of using data to learn behaviours of systems and design controllers where the end goal generally evolves around stabilization. Transfer operators i.e. Perron-Frobenius and Koopman operators play an undeniable role in advanced research of nonlinear dynamical system stabilization. These operators have been a alternate direction of how we generally approach dynamical systems, providing linear representations for even strongly nonlinear dynamics. There is tremendous benefit of acquiring a linear model of a system using these models but, there remains a challenge of infinite dimension for such models. To deal with it, we can approximate a finite dimensional matrix of these operators e.g. Koopman matrix using Extended Dynamic Mode Decomposition (EDMD) or Naturally Structured Dynamic Mode Decomposition (NSDMD). Using duality property of Koopman and P-F operators we can derive formulation for P-F matrix from Koopman matrix. Once we have a linear approximation of the system, Lyapunov measure approach can be used along with a linear programming based computational framework for stability analysis and design of almost everywhere stabilizing controller. In this work, we propose a complete structure to stabilize a system that does not have an explicit model and only requires black box input output time-series data. On a separate work, we show a set-oriented approach can be used to control and stabilize systems with known dynamics model however having stochastic parameters. Essentially, this work proposes two approaches to stabilize a nonlinear system using both of known system model with inherent uncertainty and stabilize a black box system entirely using input-output data.

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CHAPTER 1. INTRODUCTION

Dynamical systems are studied in many aspects of science, technology, economics, evolution and what not. There is fun to work with these systems, gather insights, observe trajectories but these systems come with lot of challenges which are known to the researchers in these domains. Linear systems are easier to deal with and multiple tools and algorithms exist to understand those. However, nonlinear systems are harder to approach, gather insight or analyze.

Designing controllers for dynamical systems and performing optimal stabilization are two of the most active research areas within control systems community. Lyapunov functions are used for stability analysis and control Lyapunov functions are used in the design of stabilizing feedback controllers. The goal of our research is to solve stabilization problems for nonlinear dynamical systems. In this work, specifically Transfer operator approach is used for controller design and stabilization of such systems. Transfer Perron-Frobenius and Koopman operators allow us to obtain useful insight of the system added to what we have from state space dynamics. The basic idea behind these methods is to shift the focus from the state space where the system evolution is nonlinear to measure space or space of functions where the system evolution is linear. The linearity of the transfer operator framework offers several advantages for analysis and design problems involving nonlinear systems. However, analyzing the evolution of densities and observables are the most important prerequisites for dealing with transfer operators.

1.1 Survey of Related Work

The standard approach for control system can be divided into two parts: system identification or modeling, and control design for desired performance. Stability analysis and stabilization of dynamical systems are two classical problems in control theory with applications ranging across various engineering discipline. Systematic tools exist for stability analysis and control design for

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linear systems, however, for nonlinear systems, this is still an active area of research. The introduction of linear transfer operator theoretic methods from dynamical system theory provides an opportunity to analyze and design nonlinear systems Dellnitz and Junge (1999); Mezic and Banaszuk (2004); Froyland (2001); Junge and Osinga (2004b); Dellnitz et al. (2005); Mezić (2005); Mehta and Vaidya (2005); Vaidya and Mehta (2008). These operators can be specifically used for systematic approach for the stability analysis and stabilization of nonlinear systems Lasota and Mackey (1994); Dellnitz and Junge (2002); Mezić (2005). The transfer operator theoretic methods involving Perron-Frobenius (P-F) and Koopman operators provide a linear representation of a nonlinear system by shifting the focus from the state space to the space of measures or densities and functions.

In particular, transfer operator-based methods are used for identifying steady state dynamics of the system from the invariant measure of transfer operator, identifying almost invariant sets, and coherent structures Dellnitz and Junge (2002); Froyland and Dellnitz (2003); Froyland and Padberg (2009). The spectral analysis of transfer operators are also applied for reduced order modeling of dynamical systems with applications to building systems, power grid, and fluid mechanics Mehta and Vaidya (2005); Budišić et al. (2012). Operator-theoretic methods have also been successfully applied to address design problems in control dynamical systems. Transfer operator methods can be used for almost everywhere stability verification, control design, nonlinear estimation, and for solving optimal sensor placement problem Surana and Banaszuk (2016); Raghunathan and Vaidya (2014); Vaidya and Mehta (2008); Rajaram et al. (2010); Vaidya et al. (2010); Vaidya (2007b); Sinha et al. (2016). Linear nature of the transfer P-F operator is exploited using linear programming based procedure for stability analysis and optimal control design of nonlinear systems Raghunathan and Vaidya (2014); Das et al. (2017). Lyapunov measure and control Lyapunov measure were introduced for almost everywhere stability verification and design of stabilizing feedback controller for a nonlinear system.

On the other hand, in this era of big data, there is new excitement towards developing data-driven methods for the analysis and control of complex dynamics Schmid (2010); Brunton et al.

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(2016); Kutz et al. (2016). This has led to continued interest in the data-driven approximation of Koopman and P-F operators. These data-driven methods predominantly evolve around finite-dimensional approximation of Koopman operator, dual to transfer P-F operator Rowley et al. (2009); Williams et al. (2015); Klus et al. (2015); Huang and Vaidya (2016). Spectral analysis of Koopman operator and its finite dimensional approximation constructed from time-series data has been successfully applied to address analysis problems in several applications Budisic et al. (2012); Susuki and Mezic (2011); Surana and Banaszuk (2016); Mehta and Vaidya (2005). There has also been attempt to extend their applicability for control design for nonlinear systems Kaiser et al. (2017); Peitz and Klus (2017); Korda and Mezi (2018). However, they do not exploit the real potential and linear nature of Koopman operator for control design i.e., they do not provide linear approach for control design of nonlinear system.

1.2 Main Contributions of Thesis

The main contribution of this work is to show that systematic data-driven linear methods can be developed for optimal controller design of nonlinear system exploiting the true potential of the linear operator theoretic framework. This method, relies on gathering knowledge of state space using transfer operators to obtain globally optimal stabilizing control.

This main contribution towards developing systematic data-driven control design for a nonlinear system is made possible by utilizing not only the linearity but also positivity, Markov property, and duality between Koopman and P-F operators. In particular Naturally Structured Dynamic Mode Decomposition (NSDMD) algorithm provides a data-driven approximation of Koopman and P-F operators and preserves positivity and Markov properties of these operators Huang and Vaidya (2016). We show that the NSDMD algorithm can be combined with systematic model-based transfer P-F operator approach to provide a data-driven linear programming-based method for optimal control of a nonlinear system. However, there are certain choices of basis function depending on the system dimension and number of basis function, spread of state-space and type of instability if known. We showed the true potential of the approach making use of multiple simulation of

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both discrete and continuous time systems with different dimensions and having different type of instabilities. Also, as a connected work, we showed set-oriented numerical stabilization for a system that offers a true model but has stochastic parameters.

1.3 Organization of Thesis

In chapter 2, several definitions and theorems are discussed explaining Transfer operator, Dynamic Model Decomposition and Lyapunov measure based stability analysis.

In chapter 3, We discuss our proposed approaches for optimal stabilization. This is a brief discussion containing the main ideas and steps on the algorithm. Details and Simulations are discussed in following chapters.

Chapter 4 contains additional numerical detail and the simulations for data-driven optimal control design.

Chapter 5 is self-contained with detail of set-oriented approach for stochastic stabilization.

In chapter 6, we make concluding remarks for the thesis overall.

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CHAPTER 2. TRANSFER OPERATORS, DMD, EDMD, NSDMD, LYAPUNOV MEASURE, OPTIMAL STABILIZATION

2.1 Transfer Operators and Their Properties

If a given system operates on a density as an initial condition, rather than on a single point, then successive densities are given by a linear integral operator, known as the Frobenius-Perron operator Lasota and Mackey (1994). We will briefly discuss Frobenius-Perron alternatively called Perrron-Frobenius or P-F operator, followed by another closely related operator called Koopman operator.

Consider a discrete time dynamical system

$$x_{t+1} = T(x_t) \tag{2.1}$$

where $T : X \subset \mathbb{R}^N \to X$ is assumed to be invertible and smooth diffeomorphism. Furthermore, we denote by $\mathcal{B}(X)$ the Borel-σ algebra on X and $\mathcal{M}(X)$ vector space of bounded complex valued measure on X. Associated with this discrete time dynamical systems are two linear operators namely Koopman and Perron-Frobenius (P-F) operator. These two operators are defined as follows.

Definition 1 (Perron-Frobenius Operator) $\mathbb{P}_T : \mathcal{M}(X) \to \mathcal{M}(X)$ *is given by*

$$[\mathbb{P}\mu](A) = \int_{\mathcal{X}} \delta_{T(x)}(A) d\mu(x) = \mu(T^{-1}(A))$$

$\delta_{T(x)}(A)$ *is stochastic transition function which measure the probability that point x will reach the set A in one time step under the system mapping T.*

Definition 2 (Invariant measure) *The fixed points of the P-F operator $\mathbb{P}_T$ that are additionally probability measures. Let $\bar{\mu}$ be the invariant measure then, $\bar{\mu}$ satisfies*

$$\bar{\mu}\mathbb{P} = \bar{\mu}$$

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Under the assumption that the state space X is compact it can be shown that the P-F operator admits at least one invariant measure.

Definition 3 (Koopman Operator) *Given any $h \in \mathcal{F}$, $\mathbb{U} : \mathcal{F} \to \mathcal{F}$ is defined by*

$$[\mathbb{U}h](x) = h(T(x))$$

We can state the following properties for the Koopman and Perron-Frobenius operators.

i) For $\mathcal{F} = L_2(X, \mathcal{B}, \bar{\mu})$ as the Hilbert space it is easy to see that

$$\begin{aligned} \| \mathbb{U}h \|^2 &= \int_X |h(T(x))|^2 d\bar{\mu}(x) \\ &= \int_X |h(x)|^2 d\bar{\mu}(x) = \| h \|^2 \end{aligned}$$

where we used the fact the $\bar{\mu}$ is an invariant measure. This implies that Koopman operator is unitary.

ii) For any $h \geq 0$, we have $[\mathbb{U}h](x) \geq 0$ and hence Koopman is a positive operator.

iii) If we define P-F operator act on the space of densities i.e., $L_1(X)$ and Koopman operator on space of $L_\infty(X)$ functions, then it can be shown that the P-F and Koopman operators are dual to each others as follows [1]

$$\begin{aligned} \langle \mathbb{U}f, g \rangle &= \int_X [\mathbb{U}f](x) g(x) dx \\ &= \int_X f(y) g(T^{-1}(y)) \left| \frac{dT^{-1}}{dy} \right| dy = \langle f, \mathbb{P}g \rangle \end{aligned}$$

where $f \in L_\infty(X)$ and $g \in L_1(X)$ and the P-F operator on the space of densities $L_1(X)$ is defined as follows

$$[\mathbb{P}g](x) = g(T^{-1}(x)) \left| \frac{dT^{-1}(x)}{dx} \right|$$

iv) For $g(x) \geq 0$, $[\mathbb{P}g](x) \geq 0$.

[1] with some abuse of notation we are using the same notation for the P-F operator defined on the space of measure and densities.

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v) Let $(X, \mathcal{B}, \mu)$ be the measure space where μ is a positive but not necessarily the invariant measure of $T : X \to X$, then the P-F operator $\mathbb{P} : L_1(X, \mathcal{B}, \mu) \to L_1(X, \mathcal{B}, \mu)$ satisfies following properties.

$$\int_X [\mathbb{P}g](x) d\mu(x) = \int_X g(x) d\mu(x)$$

The linearity of the P-F operator combined with the properties 2.1 (iv) and 2.1 (v), makes the P-F operator a particular case of Markov operator. This Markov property of P-F operator has significant role on its finite dimensional approximation. We will discuss it in following chapters on set-oriented numerical methods and gaussian basis function based finite dimensional approximation of P-F operator. Since $\mathbb{P}$ and $\mathbb{U}$ are unitary operators their spectrum lies on the unit circle. Given the adjoint nature of two operators, the spectrum of these operators are related. To study the connection between the spectrum of these two operators, interested readers can follow the analysis in Mezic and Banaszuk (2004) and Mehta and Vaidya (2005) (Theorem 5 and Corollary 6) for results connecting the spectrum of transfer Koopman and P-F operator both in infinite dimensional and finite dimensional setting.

2.2 Data-driven Approximation of Transfer Operators

2.2.1 Dynamic mode decomposition (DMD), Extended DMD and NSDMD

Dynamic Mode Decomposition method (DMD) has been introduced Schmid (2010) for the dynamical analysis of the fluid flow field data. DMD can be used as a computation algorithm for approximating the spectrum of Koopman operator Rowley et al. (2009). Extension of the DMD is presented in the form of Extending DMD (EDMD) Williams et al. (2015) which is more realistic when it comes to approximating the spectrum of Koopman operator for both linear and nonlinear systems. To be precise, DMD can be viewed as a special case of EDMD. Going further

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in the direction to DMD, there is Naturally Structured Dynamic Mode Decomposition (NSMD) proposed to incorporate inherent properties of Koopman and P-F operators Huang and Vaidya (2016). Having taken care of these properties NSDMD allows to approximate the operators in a more realistic fashion.

Consider snapshots of data set obtained from a discrete time dynamical system $z \to T(z)$ or from an experiment

$$\overline{X} = [x_1, x_2, \dots, x_M], \quad \overline{Y} = [y_1, y_2, \dots, y_M] \tag{2.2}$$

where $x_i \in X$ and $y_i \in X$. The two pair of data sets are assumed to be related and one is just a delayed by one step in time i.e., $y_i = T(x_i)$. Now, let $\mathcal{D} = \{\psi_1, \psi_2, \dots, \psi_K\}$ be the set of dictionary functions or observables. The dictionary functions are assumed to belong to $\psi_i \in L_2(X, \mathcal{B}, \mu) = \mathcal{G}$, where μ is some positive measure not necessarily the invariant measure of T. Let $\mathcal{G}_{\mathcal{D}}$ denote the span of $\mathcal{D}$ such that $\mathcal{G}_{\mathcal{D}} \subset \mathcal{G}$. The choice of dictionary functions are very crucial and it should be rich enough to approximate the leading eigenfunctions of Koopman operator. Define vector valued function $\mathbf{\Psi} : X \to \mathbb{C}^K$

$$\mathbf{\Psi}(\boldsymbol{x}) := \begin{bmatrix} \psi_1(x) & \psi_2(x) & \cdots & \psi_K(x) \end{bmatrix} \tag{2.3}$$

In this application, $\mathbf{\Psi}$ is the mapping from physical space to feature space. Any function $\phi, \hat{\phi} \in \mathcal{G}_{\mathcal{D}}$ can be written as

$$\phi = \sum_{k=1}^{K} a_k \psi_k = \mathbf{\Psi}^{\boldsymbol{T}} \boldsymbol{a}, \quad \hat{\phi} = \sum_{k=1}^{K} \hat{a}_k \psi_k = \mathbf{\Psi}^{\boldsymbol{T}} \hat{\boldsymbol{a}} \tag{2.4}$$

for some set of coefficients $\boldsymbol{a}, \hat{\boldsymbol{a}} \in \mathbb{C}^K$. Let

$$\hat{\phi}(x) = [\mathbb{U}\phi](x) + r,$$

where $r \in \mathcal{G}$ is a residual function that appears because $\mathcal{G}_{\mathcal{D}}$ is not necessarily invariant to the action of the Koopman operator. To find the optimal mapping which can minimize this residual, let $\mathbf{K}$ be the finite dimensional approximation of the Koopman operator.

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Then the matrix $\mathbf{K}$ is obtained as a solution of least square problem as follows

$$\min_{\mathbf{K}} \| \mathbf{GK} - \mathbf{A} \|_F \tag{2.5}$$

$$\mathbf{G} = \frac{1}{M}\sum_{m=1}^{M} \mathbf{\Psi}(x_m)^\top \mathbf{\Psi}(x_m)$$
$$\mathbf{A} = \frac{1}{M}\sum_{m=1}^{M} \mathbf{\Psi}(x_m)^\top \mathbf{\Psi}(y_m), \tag{2.6}$$

with $\mathbf{K}, \mathbf{G}, \mathbf{A} \in \mathbb{C}^{K\times K}$. The optimization problem (3.2) can be solved explicitly to obtain following solution for the matrix $\mathbf{K}$

$$\mathbf{K}_{EDMD} = \mathbf{G}^\dagger \mathbf{A} \tag{2.7}$$

where $\mathbf{G}^\dagger$ is the psedoinverse of matrix $\mathbf{G}$. Hence, under the assumption that the leading Koopman eigenfunctions are nearly contained within $\mathcal{G}_\mathcal{D}$, the subspace spanned by the elements of $\mathcal{D}$. The eigenvalues of $\mathbf{K}$ are the EDMD approximation of Koopman eigenvalues. The right eigenvectors of $\mathbf{K}$ generate the approximation of the eigenfunctions in (2.8). In particular, the approximation of Koopman eigenfunction is given by

$$\phi_j = \mathbf{\Psi} v_j \tag{2.8}$$

where v_j is the j-th right eigenvector of $\mathbf{K}$, ϕ_j is the eigenfunction approximation of Koopman operator associated with j-th eigenvalue.

DMD is a particular case of EDMD, and it corresponds to the case where the dictionary functions are chosen to be equal to $\mathcal{D} = \{e_1^\top, \ldots, e_K^\top\}$, where $e_i \in \mathbb{R}^N$ is a unit vector with 1 at i^{th} position and zero elsewhere. With this choice of dictionary function, it can be shown the approximation of the Koopman operator using DMD approach can be written as

$$\mathbf{K}_{DMD} = \overline{Y}\,\overline{X}^\dagger,$$

where $\overline{X}$ and $\overline{Y}$ are dataset as defined in (2.2).

NSDMD is EDMD with added constraints to preserve positivity and markov properties of the transfer operators. Here, dictionary functions are same as EDMD. We will explain some properties of PF and Koopman Operator and outline the NSDMD algorithm.

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The Koopman operator corresponding to dynamical system (2.14) is defined as

$$[\mathbb{U}h](x) = h(F(x)),$$

where $h \in \mathcal{C}^0(X)$. The Koopman and P-F operators are dual to each other and the duality is expressed as follows [2]

$$\begin{aligned} \langle \mathbb{U}h, g \rangle &= \int_X [\mathbb{U}h](x) g(x) dx \\ &= \int_X h(x) [\mathbb{P}g](x) dx = \langle h, \mathbb{P}g \rangle , \end{aligned} \tag{2.9}$$

where $h \in \mathcal{L}_\infty(X)$ and $g \in \mathcal{L}_1(X)$ and the P-F operator on the space of densities are defined as follows:

$$[\mathbb{P}g](x) = g(F^{-1}(x)) |\frac{dF^{-1}(x)}{dx}|$$

Furthermore, these two operators also satisfy positivity property i.e., for any $h \geq 0$ and $g \geq 0$, we have $\mathbb{U}h \geq 0$ and $\mathbb{P}g \geq 0$. Another important property the P-F operator satisfies is the Markov property

$$\int_X [\mathbb{P}g](x) d\mu(x) = \int_X g(x) d\mu(x),$$

where $\mathbb{P} : \mathcal{L}_1(X, \mu) \to \mathcal{L}_1(X, \mu)$ and μ is not necessarily invariant probability measure. The NSDMD algorithm approximate Koopman operator while preserving the positivity property. Furthermore, the duality between the Koopman and P-F operator combined with the Markov property of the P-F operator is exploited to provide data-driven approximation of P-F operator from the Koopman operator. Hence the NSDMD algorithm can be viewed as Extended Dynamic Mode Decomposition (EDMD) with added constraints to ensure positivity and Markov property Williams et al. (2015).

Assumption 4 *We assume that* $\psi_j(x) \geq 0$ *for* $j = 1, \ldots, K$ *and define*

$$[\Lambda]_{ij} = \int_X \psi_i(x) \psi_j(x) dx. \tag{2.10}$$

[2] With some abuse of notation we are using the same notation to define the P-F operator acting on the space of functions and measures

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Under Assumption 4, the finite dimensional approximation of Koopman operator $K \in \mathbb{R}^{K\times K}$, and P-F operator $P \in \mathbb{R}^{K\times K}$, can be formulated as following optimization problem

$$\min_{K} \parallel \mathbf{G}K - \mathbf{A} \parallel_F \tag{2.11}$$
$$\text{s.t. } K_{ij} \geq 0, \quad \text{(Koopman positive constraints)}$$
$$[\Lambda K \Lambda^{-1}]_{ij} \geq 0, \quad \text{(P} - \text{F positive constraints)}$$
$$\Lambda K \Lambda^{-1}\mathbf{1} = \mathbf{1}, \quad \text{(P} - \text{F Markov constraints)}$$

where $\mathbf{G}$ and $\mathbf{A}$ are defined as follows:

$$\mathbf{G} = \frac{1}{M}\sum_{m=1}^{L} \mathbf{\Psi}(x_m)^\top \mathbf{\Psi}(x_m)$$
$$\mathbf{A} = \frac{1}{M}\sum_{m=1}^{L} \mathbf{\Psi}(x_m)^\top \mathbf{\Psi}(y_m). \tag{2.12}$$

and $\mathbf{1}$ is the vector of all ones. The P-F operator P is given by $P = \Lambda^{-1}K^\top\Lambda$.

The matrix Λ in Eq. (2.10) can be computed explicitly.

2.3 Set-oriented Numerical Approach for Controller Design

Set-oriented numerical methods are primarily developed for the finite dimensional approximation of the Perron-Frobenius operator for the case where system dynamics are known Dellnitz and Junge (2002); Dellnitz et al. (2001). However, these algorithms can be modified or extended to the case where system information is available in the form of time series data. The basic idea behind set-oriented numerics is to partition the state space, X, into the disjoint set of boxes D_i such that $X = \cup_{i=1}^{N} D_i$. Consider a finite partition $X^{'} = \{D_1, \ldots, D_K\}$. Now, instead of a Borel σ-algebra, consider a σ-algebra of all possible subsets of X. A real-valued measure μ_j is defined by attributing to each element D_j a real number. This allows one to identify the associated measure space with a finite-dimensional real vector space $\mathbb{R}^K$. A given mapping $T : X \to X$ defines a stochastic

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transition function $\delta_{T(x)}(\cdot)$. This function can be used to obtain a coarser representation of P-F operator denoted by $\mathbf{P}' : \mathbb{R}^{K\times K} \to \mathbb{R}^{K\times K}$ as follows: For $\mu^{'} = (\mu^{'}_1, \ldots, \mu^{'}_K)$ we define a measure on X as

$$d\mu(x) = \sum_{k=1}^{K} \mu^{'}_k \chi_{D_k}(x) \frac{dm(x)}{m(D_k)}$$

where $\chi_{D_k}(x)$ is the indicator function of D_k and dm is the Lebesgue measure. The finite dimensional approximation of the P-F matrix, $\mathbf{P}'$, can now be obtained as follows:

$$\nu'_i = [\mathbf{P}'\mu'](D_i) = \sum_{j=1}^{K} \int_{D_j} \delta_{T(x)}(D_i)\mu'_j \frac{dm(x)}{m(D_j)}$$

$$= \sum_{j=1}^{K} \mu'_k \mathbf{P}'_{ij} \tag{2.13}$$

where

$$\mathbf{P}'_{ij} = \frac{m(T^{-1}(D_j) \cap D_i)}{m(D_j)}$$

The resulting matrix $\mathbf{P}'$ is a Markov matrix and is row stochastic if we consider state μ' to be a row vector multiplying from the left of P. The individual entries of this Markov matrix can be obtained by Monte-Carlo approach by running simulation over short time interval starting from different initial conditions. Typically individual boxes D_i will be populated with M uniformly distributed initial conditions. The entry $\mathbf{P}_{ij}$ is then approximated by fraction of initial conditions that are in box D_j in one forward iteration of the mapping T. The Monte Carlo based approach can be extended for computation of the P-F transfer operator from time series data. Let $\{x_0, T(x_0), \ldots, T^{N-1}(x_0)\}$ be the time series data set. The number of initial conditions in box i is then given by

$$\sum_{k=0}^{N-1} \chi_i(T^k(x_0))$$

where χ_i is the indicator function of box i. The (i, j) entry for P-F matrix $\mathbf{P}'_{ij}$ is then given by the fraction of these initial conditions from box i that ends up in box j after one iterate of time and is given by following formula.

$$\mathbf{P}'_{ij} = \frac{1}{\sum_{k=0}^{N-1} \chi_i(T^k(x_0))} \sum_{k=0}^{N-1} \chi_i(T^k(x_0))\chi_j(T^{k+1}(x_0)).$$

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2.4 Lyapunov Measure and Stability

In this section we provide brief overview of the application of linear transfer P-F operator framework for almost everywhere stability analysis and optimal stabilization of nonlinear system using Lyapunov measure Vaidya and Mehta (2008); Vaidya et al. (2010); Raghunathan and Vaidya (2014).

2.4.1 Lyapunov measure for stabilization

Consider the discrete-time dynamical systems of the form,

$$x_{n+1} = F(x_n), \tag{2.14}$$

where $F : X \to X$ is assumed to be continuous with $X \subset \mathbb{R}^q$, a compact set. We denote $\mathcal{B}(X)$ as the Borel-σ algebra on X and $\mathcal{M}(X)$ as the vector space of a real valued measure on $\mathcal{B}(X)$. The mapping F is assumed to be nonsingular with respect to the Lebesgue measure ℓ, i.e., $\ell(F^{-1}(B)) = 0$, for all sets $B \in \mathcal{B}(X)$, such that $\ell(B) = 0$. In this paper, we are interested in data-driven optimal stabilization of an attractor set defined as follows:

Definition 5 (Attractor set) *A set $\mathcal{A} \subset X$ is said to be forward invariant under F, if $F(\mathcal{A}) = \mathcal{A}$. A closed forward invariant set $\mathcal{A}$ is said to be an attractor set, if there exists a neighborhood $V \subset X$ of $\mathcal{A}$, such that $\omega(x) \subset \mathcal{A}$ for all $x \in V$, where $\omega(x)$ is the ω limit set of x.*

Remark 6 *We will use the notation $U(\epsilon)$ to denote the $\epsilon > 0$ neighborhood of the attractor set $\mathcal{A}$ and $m \in \mathcal{M}(X)$, a finite measure absolutely continuous with respect to Lebesgue.*

Definition 7 (a.e. stable with geometric decay) *The attractor set $\mathcal{A} \subset X$ for a dynamical system (2.14) is said to be almost everywhere (a.e.) stable with geometric decay with respect to some finite measure $m \in \mathcal{M}(X)$, if given any $\epsilon > 0$, there exists $M(\epsilon) < \infty$ and $\beta < 1$, such that $m\{x \in \mathcal{A}^c : F^n(x) \in X \setminus U(\epsilon)\} < M(\epsilon)\beta^n$.*

The above set-theoretic notion of a.e. stability was introduced and verified by using the linear transfer operator framework Vaidya and Mehta (2008). For the discrete time dynamical system

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(2.14), the linear transfer Perron Frobenius (P-F) operator denoted by $\mathbb{P}_F : \mathcal{M}(X) \to \mathcal{M}(X)$ is given by,

$$[\mathbb{P}_F \mu](B) = \int_X \chi_B(F(x)) d\mu(x) = \mu(F^{-1}(B)), \tag{2.15}$$

where $\chi_B(x)$ is the indicator function supported on the set $B \in \mathcal{B}(X)$ and $F^{-1}(B)$ is the inverse image of set B Lasota and Mackey (1994). We define a sub-stochastic operator as a restriction of the P-F operator on the complement of the attractor set as follows:

$$[\mathbb{P}_F^1 \mu](B) := \int_{\mathcal{A}^c} \chi_B(F(x)) d\mu(x), \tag{2.16}$$

for any set $B \in \mathcal{B}(\mathcal{A}^c)$ and $\mu \in \mathcal{M}(\mathcal{A}^c)$. The condition for the a.e. stability of an attractor set $\mathcal{A}$ with respect to some finite measure m is defined in terms of the existence of the *Lyapunov measure* $\bar{\mu}$, defined as follows Vaidya and Mehta (2008).

Definition 8 (Lyapunov measure) *The Lyapunov measure is defined as any non-negative measure $\bar{\mu}$, finite outside $U(\epsilon)$ (see Remark 6), and satisfies the following inequality, $[\mathbb{P}_F^1 \bar{\mu}](B) < \gamma^{-1} \bar{\mu}(B)$, for some $\gamma \geq 1$ and all sets $B \in \mathcal{B}(X \setminus U(\epsilon))$, such that $m(B) > 0$.*

The following theorem provides the condition for a.e. stability with geometric decay Vaidya (2007a).

Theorem 9 *An attractor set $\mathcal{A}$ for the dynamical system (2.14) is a.e. stable with geometric decay with respect to finite measure m, if and only if for all $\epsilon > 0$ there exists a non-negative measure $\bar{\mu}$, which is finite on $\mathcal{B}(X \setminus U(\epsilon))$ and satisfies*

$$\gamma [\mathbb{P}_F^1 \bar{\mu}](B) - \bar{\mu}(B) = -m(B), \tag{2.17}$$

for all measurable sets $B \subset X \setminus U(\epsilon)$ and for some $\gamma > 1$ and where the geometric decay rate is given by $\beta \leq \frac{1}{\gamma} < 1$

Proof 10 *We refer readers to Theorem 9 from Vaidya (2007a) for the proof.*

Stabilization of the attractor set is posed as a co-design problem of jointly obtaining the control Lyapunov measure $\bar{\mu}$ and the control P-F operator $\mathbb{P}_C$ Vaidya et al. (2010).

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In the following section we explain how the stabilization framework using Lyapunov measure can be extended to optimization stabilization using Lyapunov measure.

2.4.2 Optimal stabilization

The basic idea behind the optimal stabilization is to augment the control Lyapunov measure equation with a cost function so that the attractor set $\mathcal{A}$ is stabilized while minimizing a certain cost.

We consider the following cost function.

$$\mathcal{C}_C(B) = \int_B \sum_{n=0}^{\infty} \gamma^n G \circ C(x_n) dm(x), \tag{2.18}$$

where $x_0 = x$, the cost function $G : Y \to \mathbb{R}$ is assumed a continuous non-negative real-valued function, such that $G(\mathcal{A}, 0) = 0$, $x_{n+1} = T \circ C(x_n)$, and $0 < \gamma < \frac{1}{\beta}$. Under the assumption that the controller mapping C renders the attractor set a.e. stable with a geometric decay rate, $\beta < \frac{1}{\gamma}$, the cost function (2.18) is finite. In the following we will use the notion of the scalar product between continuous function $h \in \mathcal{C}^0(X)$ and measure $\mu \in \mathcal{M}(X)$ as $\langle h, \mu \rangle_X := \int_X h(x) d\mu(x)$ Lasota and Mackey (1994). The following theorem proves the cost of stabilization of the set $\mathcal{A}$ as given in Eq. (2.18) can be expressed using the control Lyapunov measure equation.

Theorem 11 *Let the controller mapping $C(x) = (x, K(x))$, be such that the attractor set $\mathcal{A}$ for the feedback control system $T \circ C : X \to X$ is a.e. stable with geometric decay rate $\beta < 1$. Then, the cost function (2.18) is well defined for $\gamma < \frac{1}{\beta}$ and, furthermore, the cost of stabilization of the attractor set $\mathcal{A}$ with respect to Lebesgue almost every initial condition starting from set $B \in \mathcal{B}(X_1)$ can be expressed as follows:*

$$\begin{aligned} \mathcal{C}_C(B) &= \textstyle\int_B \sum_{n=0}^{\infty} \gamma^n G \circ C(x_n) dm(x) \\ &= \textstyle\int_{\mathcal{A}^c \times U} G(y) d[\mathbb{P}_C^1 \bar{\mu}_B](y) = \left\langle G, \mathbb{P}_C^1 \bar{\mu}_B \right\rangle_{\mathcal{A}^c \times U}, \end{aligned} \tag{2.19}$$

where $x_0 = x$ and $\bar{\mu}_B$ is the solution of the following control Lyapunov measure equation,

$$\gamma \mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}_B(D) - \bar{\mu}_B(D) = -m_B(D), \tag{2.20}$$

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for all $D \in \mathcal{B}(X_1)$ *and where* $m_B(\cdot) := m(B \cap \cdot)$ *is a finite measure supported on the set* $B \in \mathcal{B}(X_1)$.

Proof 12 *Refer to Raghunathan and Vaidya (2012) (Theorem 24) for the proof.*

By appropriately selecting the measure on the right-hand side of the control Lyapunov measure equation (2.20) (i.e., m_B), stabilization of the attractor set with respect to a.e. initial conditions starting from a particular set can be studied. The minimum cost of stabilization is defined as the minimum over all a.e. stabilizing controller mappings C with a geometric decay as follows:

$$\mathcal{C}^*(B) = \min_{C} \mathcal{C}_C(B). \tag{2.21}$$

Using (5.12) and (2.20) the infinite dimensional linear program for optimal stabilization can be written as follows. We first define the projection map, $P_1 : \mathcal{A}^c \times U \to \mathcal{A}^c$ as: $P_1(x, u) = x$, and denote the P-F operator corresponding to P_1 as $\mathbb{P}_{P_1} : \mathcal{M}(\mathcal{A}^c \times U) \to \mathcal{M}(\mathcal{A}^c)$, which can be written as $[\mathbb{P}^1_{P_1}\theta](D) = \int_{\mathcal{A}^c \times U} \chi_D(P_1(y))d\theta(y) = \int_{D\times U} d\theta(y) = \mu(D)$. Using this definition of projection mapping P_1 and the corresponding P-F operator, we can write the linear program for the optimal stabilization of set B with unknown variable θ as follows:

$$\begin{aligned} &\min_{\theta \geq 0} \langle G, \theta \rangle_{\mathcal{A}^c \times U}, \\ \text{s.t. } \; &\gamma[\mathbb{P}^1_T\theta](D) - [\mathbb{P}^1_{P_1}\theta](D) = -m_B(D), \end{aligned} \tag{2.22}$$

for $D \in \mathcal{B}(X_1)$.

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CHAPTER 3. PROPOSED APPROACHES

3.1 System

A control system in practice generally consists of a system black box, multiple sensors to acquire measurements of current states or outputs, and the controller, which decides the next control action to take based on the outputs and desired target. In many cases, one might not have access to the original states x_i, rather will have access to the certain outputs y. Controller will try to reach the reference r (for stabilization, these are the fixed or equilibrium point of the system), therefore will take certain control actions depending on current measured outputs y.

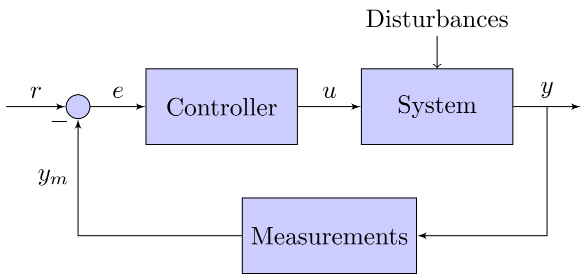

Figure 3.1: Control System

3.2 Data-driven Optimal Stabilization

Assume we have a black box model of the system and we can obtain input output time-series data. The end goal is to find a stabilizing controller for the system. Consider the system to be discrete time for now. The main steps we propose to reach the goal.

- Approximate a system model from data using Koopman operator.
- Approximate the Perron-Frobenius operator using duality property

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- Use linear programming based approach for control design and optimal stabilization

To expand on these main steps and give a complete structure of the algorithm used for the work, we summarize the algorithm for data-driven control as the following

1) **Data acquisition:** Obtain input output time-series data from the dynamical system. These are essentially evolution of states which will be used to obtain a evolution of observables in following steps.

$$X = [x_1, x_2, \dots, x_L]$$

$$Y = [y_1, y_2, \dots, y_L]$$

Where, $x_i \in \mathbb{R}^n$ are states for the discrete time transformation $T : X \subset \mathbb{R}^N \to X$. Again, X and Y are related by just a one time step delay $y_i = T(x_i)$.

2) **Observable Space:** Introduce appropriate basis functions for defining the observable space. These basis functions should be able to represent the function space where the evolution will be linear. Moreover, considering the positivity property of Perron-Frobenius operator, we chose a basis function which is positive. For the scope of control design work, it has been observed and studied that gaussian radial basis functions are a good candidate for such representation as they are nonzero nonnegative functions everywhere in the state space.

$$\mathbf{\Psi}(x) = [\psi_1(x), \dots, \psi_N(x)]^\top$$

$$\psi_i(x) = e^{\frac{(x-x_i)^2}{\sigma^2}} \tag{3.1}$$

where $x \in \mathbb{R}^n$ are the states and $x_i \in \mathbb{R}^n$ are centers of basis functions.

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Parameters for radial basis functions:

Number of basis functions, N

Center of basis functions, x_i

Spread of basis σ

3) **Approximate Koopman Operator:** Solve the Optimization problem either using EDMD or NSDMD.

For EDMD, the matrix $\mathbf{K}$ is obtained as a solution of least square problem as follows

$$\min_{\mathbf{K}} \| \mathbf{GK} - \mathbf{A} \|_F \tag{3.2}$$

$$\begin{aligned} \mathbf{G} &= \frac{1}{L} \sum_{m=1}^{L} \mathbf{\Psi}(x_m)^\top \mathbf{\Psi}(x_m) \\ \mathbf{A} &= \frac{1}{L} \sum_{m=1}^{L} \mathbf{\Psi}(x_m)^\top \mathbf{\Psi}(y_m), \end{aligned} \tag{3.3}$$

with $\mathbf{K}, \mathbf{G}, \mathbf{A} \in \mathbb{C}^{K \times K}$. Solution to this optimization is

$$\mathbf{K}_{EDMD} = \mathbf{G}^\dagger \mathbf{A} \tag{3.4}$$

For NSDMD, we compute Λ from the inner products of the basis functions assuming that $\psi_j(x) \geq 0$ for $j = 1, \ldots, N$ and define

$$[\Lambda]_{ij} = \int_X \psi_i(x)\psi_j(x)dx. \tag{3.5}$$

Using Kernel trick for radial basis functions, it can be shown that

$$\Lambda_{ij} = (\frac{\pi\sigma^2}{2})^{N/2} \exp{(-\frac{\|x_i - x_j\|^2}{2\sigma^2})} \tag{3.6}$$

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we solve the following optimization

$$\min_{K} \parallel \mathbf{G}K - \mathbf{A} \parallel_F \tag{3.7}$$
$$\text{s.t. } K_{ij} \geq 0, \quad \text{(Koopman positive constraints)}$$
$$[\Lambda K \Lambda^{-1}]_{ij} \geq 0, \quad \text{(P} - \text{F positive constraints)}$$
$$\Lambda K \Lambda^{-1}\mathbf{1} = \mathbf{1}, \quad \text{(P} - \text{F Markov constraints)}$$

4) **Approximate PF Operator:** Use duality of the PF and Koopman Operator to approximate a duality formula between the PF and Koopman matrices. Since, we chose gaussian radial basis functions, which are positive however not non-overlapping, we need to formulate inner products of the basis functions. We used Λ for deriving the duality using eq. 3.5.

$$P = \Lambda^{-1} K^{\top} \Lambda$$

We can use an exact formula for obtaining Λ.

5) **Disintegrating PF matrix:** From the PF matrices, P we reduce 1 row and 1 columns to obtained reduced PF matrix P^1 for feeding to the linear programming based controller design. This is done by locating the cell which contains the equilibrium or basis function which has maximum impact on observables for the equilibrium.

$$\mathbb{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \dots & P_{1N} \\ P_{21} & P_{22} & P_{23} & \dots & P_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ P_{N1} & P_{N2} & P_{N3} & \dots & P_{NN} \end{bmatrix}$$

Assume in our formulation the first cell contains the equilibrium or the first basis function center is nearest to the equilibrium. So, we will remove the 1^{st} row and 1^{st} column from PF matrix to obtain the intended sub-stochastic PF matrix $\mathbb{P}_1$.

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$$\mathbb{P}_1 = \begin{bmatrix} P_{22} & P_{23} & P_{24} & \dots & P_{2N} \\ P_{32} & P_{33} & P_{34} & \dots & P_{3N} \\ \dots & \dots & \dots & \dots & \dots \\ P_{N2} & P_{N3} & P_{N4} & \dots & P_{NN} \end{bmatrix}$$

If the equilibrium is located nearest to j^{th} rbf basis function center, we remove j^{th} row and j^{th} column from $\mathbb{P}$ to obtain $\mathbb{P}^1$.

6) **Solve Linear Program for Stochastic Control:** Formulate Linear Program using the PF matrices $\mathbb{P}^1_a$. We have obtained multiple PF matrices, one for each control action $u_i \in U_{adm}$. Approximate the stochastic controls by solving the following linear program.

$$\min_{\bar{\theta}_a \geq 0} \sum_{a=1}^{M} (G^1_a)^\top \bar{\theta}_a$$

$$\text{s.t. } \sum_{a=1}^{M} [\mathbb{I} - \gamma(\mathbb{P}^1_a)^\top]\bar{\theta}_a - m = 0$$

where, $\mathbb{P}^1_a$ corresponds to sub-stochastic PF matrix for the a^{th} control action. $\bar{\theta}_a \in \mathbb{R}^N$ is the vector containing stochastic control action weights or probabilities for each basis function. $\bar{\theta}^j_a$ corresponds to weight of choosing a^{th} control action for the j^{th} basis function. If we were to use set oriented approach these control action would represent the probability of choosing that particular action while in that cell or set.

7) **Obtain Deterministic Control:** From the stochastic control vectors obtained in step 6 can not be used directly in control design as it gives merely probabilities or weights of choosing actions for particular cells or basis functions. However, we make the approximation by choosing the control action having maximum probability or weight for a particular cell or basis function. One could use alternate approaches of their choice for doing the stochastic to deterministic approximation.

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Define optimal control for j^{th} cell or basis function as the following

$$\bar{\theta}^j_{a*} = \max\{\bar{\theta}^j_1, \dots, \bar{\theta}^j_M\}$$

For j^{th} cell or basis, we chose the action which resulted maximum probability in the optimization problem.

8) **Optimal Feedback Control:** We store the deterministic control actions corresponding to each cell or basis function. Final step is to choose control actions online for a particular state. For a state x, the scalar observable would be $\psi(x)$, and we use that to define the optimal control using weighted average from each of the basis functions. Therefore, optimal feedback control to close the loop and stabilize

$$k(x) = \sum_{j=1}^{K} u^{a(j)} \psi_j(x)$$

3.3 Set-oriented Stabilization from Known Dynamics Model with Stochastic Parameters

This approach is similar to what we discussion in section 3.2 with the exception that we no longer need to learn the model as model is known, however suffers complexity and challenges due to the addition of stochastic parameters. Therefore, for better readability of the topic, we discussed this section entirely as a different chapter 5, which again was published as a paper in 2017 American Control Conference.

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CHAPTER 4. NUMERICAL APPROXIMATION AND SIMULATION RESULTS

4.1 Finite Dimensional Approximation of Linear Program for Optimal Stabilization

We have outlined the approach in section 3.2, however for the finite dimensional approximation of the linear program, one might find this discussion particularly useful before jumping to the simulations.

The finite dimensional approximation of the P-F operator can be used in the finite dimensional approximation of the linear program in Eq. (3.2) for optimal stabilization. Towards this goal, we first discretize the control set U. The control input is quantized and assumed to take only finitely many control values from the quantized set $\mathcal{U}_M = \{u^1, \ldots, u^a, \ldots, u^M\}$, where $u^a \in \mathbb{R}^d$. For each fixed value of control input $u = u^a$, time-series data $\{x_1^a, \ldots, x_L^a\}$ for $a = 1, \ldots, M$ is generated and the finite dimensional approximation of the P-F operator is constructed using the NSDMD algorithm outlined in section 3.2. We denote the P-F operator approximated for fixed value of control input $u = u^a$ as P_a. For the finite dimensional approximation of the infinite dimensional linear program we need to approximate the cost function G and the measure θ. The centers for the Gaussian radial basis function are generated using K-mean clustering on data set generated from uncontrolled dynamical system. Let x_ℓ^* for $\ell = 1, \ldots, K$ be the centers of the Gaussian radial basis functions. The finite dimensional approximation of the cost function is then expressed as $G(x_\ell^*, u^a)$ for $\ell = 1, \ldots, K$ and $a = 1, \ldots, M$. Let $G_a = [G(x_1^*, u^a), \ldots, G(x_K^*, u^a)]^\top \in \mathbb{R}^K$ and $\theta_a \in \mathbb{R}^K$ be the finite dimensional approximation of measure θ on $X \times U$. The matrix representation of θ has K rows and M columns i.e., $\theta \in \mathbb{R}^{K\times M}$ [1]. The (j, a) entry of θ is denoted by θ_a^j and we use the notation θ_a and θ^j for the a^{th} column and j^{th} row of θ respectively.

[1] With some abuse of notations we are denoting both the infinite and finite dimensional representation of θ with same notation.

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Without loss of generality, we assume that the dictionary function $\psi_1(x)$ with center at x_1^* is supported on the equilibrium point or the attractor set that we want to stabilize. Under this assumption, let $P_a^1 \in \mathbb{R}^{(K-1)\times(K-1)}$ be the P-F matrix obtained from P_a after deleting the first row and first column. Similarly, let $\bar{\theta} \in \mathbb{R}^{(K-1)\times M}$ be the matrix obtained from θ after deleting the first row. $G_a^1 = [G(x_2^*, u^a), \ldots, G(x_K^*, u^a)]^\top \in \mathbb{R}^{K-1}$ is the vector obtained by deleting the first entry from vector G_a. The finite dimensional approximation of the infinite dimensional linear program (3.2) can then be written as follows:

$$\begin{aligned} \min_{\bar{\theta}_a \geq 0} \;\; & \textstyle\sum_{a=1}^{M} (G_a^1)^\top \bar{\theta}_a, \\ \text{s.t.} \;\; & \gamma \textstyle\sum_{a=1}^{M} (P_a^1)^\top \bar{\theta}_a - \sum_{a=1}^{M} \bar{\theta}_a = -m, \; \sum_a \bar{\theta}_a = \mathbf{1}, \end{aligned} \tag{4.1}$$

where $\mathbf{1}$ is a vector of all ones and inequality $\bar{\theta}_a \geq 0$ is element-wise. The optimization problem (4.1) is a finite dimensional linear program in terms of variable θ_a^1. The solution to the optimization problem in general lead to a stochastic vector $\bar{\theta}^j$. The row vector $\bar{\theta}^j$ has a physical significance. In particular, $\bar{\theta}_a^j$ determines the probability of choosing the control action a with state corresponding to the dictionary function $\psi_j(x)$. But, we are interested in determining deterministic control action i.e.,

$$\bar{\theta}_a^j = 1 \text{ for exactly one } a \in \{1, \ldots, M\}.$$

However, introducing this binary constraints on the entries of $\bar{\theta}^j$ in the optimization problem (4.1) will lead to non-convex formulation which is difficult to solve. Again, we know that deterministic control action can be obtained from stochastic $\bar{\theta}^j$ vector Raghunathan and Vaidya (2014). In particular, following choice of deterministic feedback control can be made from stochastic $\bar{\theta}^j$. Let $\bar{\theta}_{a*}^j = \max\{\bar{\theta}_1^j, \ldots, \bar{\theta}_M^j\}$ i.e. $a*$ is the index corresponding to the maximum entry from the vector $\bar{\theta}^j$. Then the optimal deterministic feedback control is given by

$$u^{a(j)} = u^{a*}.$$

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At this point, the optimal feedback control $k(x)$ is given by the following formula

$$k(x) = \sum_{j=1}^{K} u^{a(j)} \psi_j(x). \tag{4.2}$$

Alternatively, one could use the nearest basis function center to the state to obtain particular control value for the state.

4.2 Simulation Results

In this section we provide results of the data-driven optimal stabilization algorithm applied into one-dimensional and two-dimensional continuous and discrete time nonlinear systems. These systems have stability of different kinds starting for single fixed point, limit cycles to periodic orbits. Results are obtained using YALMIP with GUROBI solver coded in MATLAB.

Cubic Logistic Map

Controlled equation for cubic logistic map is given as:

$$x_{n+1} = \lambda x_n - x_n^3 + u_n \tag{4.3}$$

where $x_n \in [-1.6, 1.6]$ is the state, u_n is the control input and we chose parameter $\lambda = 2.3$. Let, control input space is quantized to $[-0.2 : 0.02 : 0.2]$. For the finite dimensional approximation of the P-F operator, we chose 200 Gaussian radial basis functions as dictionary function with $\sigma = 0.008$. The cost function is assumed to be $x^2 + u^2$.

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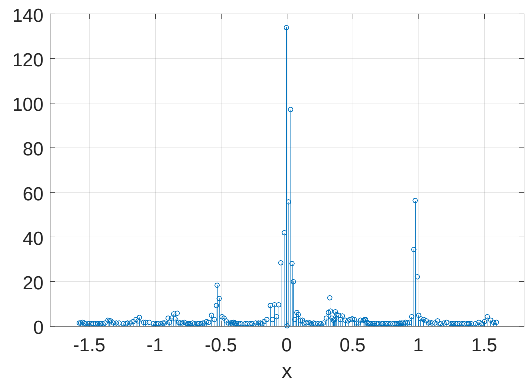

Figure 4.1: Lyapunov Measure for Cubic Logistic Map

In Fig. 4.1, we provide the Lyapunov measure plot verifying the stability of the closed loop system. Fig. 4.2 shows two sample trajectories for the open loop and closed loop logistic maps. We observe that closed-loop trajectories are perfectly stabilized to the only equilibrium point at origin within few time steps.

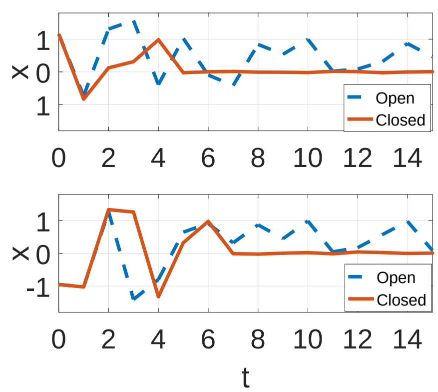

Figure 4.2: Cubic Logistic Map: Open loop and closed loop trajectory

Van der Pol Oscillator

Our second example is that of Van der pol oscillator.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2)x_2 - x_1 - u \end{aligned} \tag{4.4}$$

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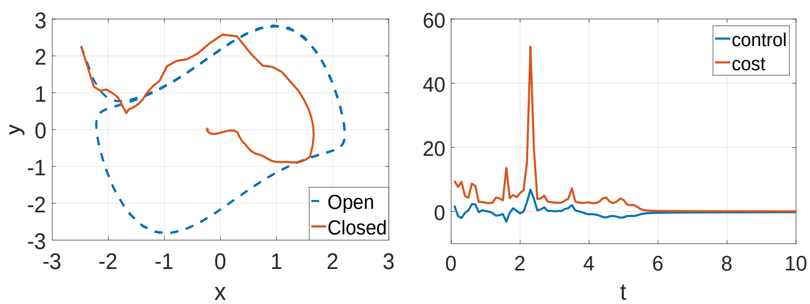
The open loop system has stable limit cycle and unstable equilibrium point at the origin. The objective is to optimally stabilize the unstable equilibrium point at the origin. The control set is discretized to $\mathcal{U} = \{-16, -14, -12..., 12, 14, 16\}$. For finite dimension approximation of the transfer operator we use 200 dictionary functions with $\sigma = 0.2$ and $X = [-5, 5] \times [-5, 5]$. In Figs. 4.3 and 4.4, we show the sample open loop and closed loop trajectories along with the plots for the optimal control inputs and optimal cost. For this particular set of data, we see the trajectories get stabilized to a point close to the equilibrium at origin. This deviation happens due to the interpolation in equation 4.2. Considering the fact, we declare a point stabilized if it reaches ϵ neighborhood of an equilibrium and remains there for the rest of the time. Choosing different number of basis functions, N or a different σ, one should be able to reach the target at origin.



(a) (b)

Figure 4.3: Vanderpol Oscillator, sample trajectory 1: a) Open loop and closed loop trajectory; b) Optimal control and cost function.

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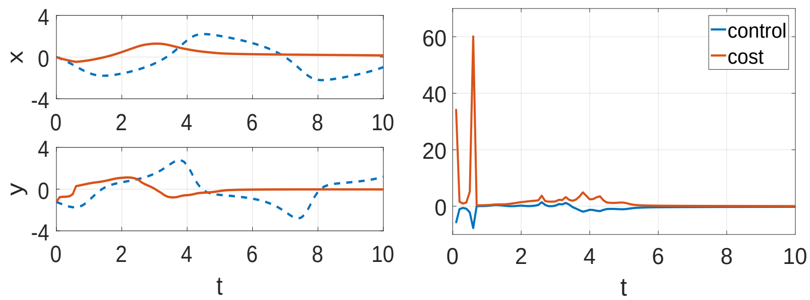

(a) (b)

Figure 4.4: Vanderpol Oscillator, sample trajectory 2: a) Open loop and closed loop trajectory; b) Optimal control and cost function.

Similarly, we simulate all 5000 initial conditions and found out 4667 of them got stabilized (93.34%) within 200 steps. We observed that most of the trajectories which could not be stabilized, were initially either in the left most or the right most edges of our phase space. Control values in those regions were not good enough to optimally stabilize them. This happens because the dictionary functions can only cover the interior of the phase space as we designed the dictionary function centers using k-means. If any trajectory ever falls out of that region, that has possibility of not getting properly mapped control values because, interpolated control values in those parts are near to zero, which may not be able to get them back to the phase space. Fig. 4.5 shows the plot of all the initial points whether they were stabilized (yellow) or remained far from the equilibrium at origin even after 200 steps or 20s simulation (blue). Again, for this example we declare stabilization if the trajectories reach the circle centered at the origin with radius 0.2 and remains there for the rest of its time. This is consistent with $\sigma = 0.2$ that we chose for the PF approximation from data.

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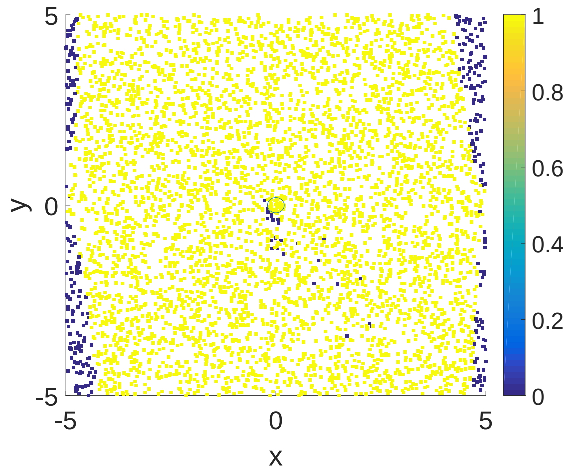

Figure 4.5: Successful Stabilization of initial conditions

Duffing Oscillator

The control of duffing oscillator is described by following equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (x_1 - x_1^3) - 0.5x_2 + u \tag{4.5}$$

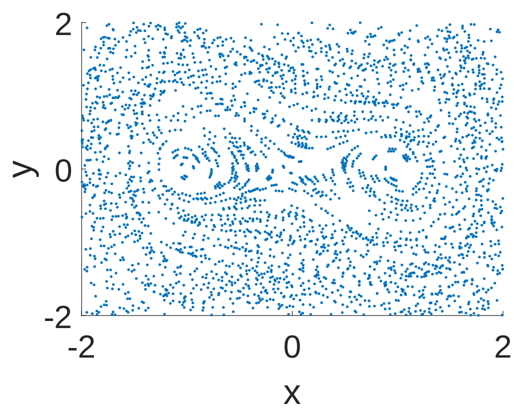

Figure 4.6: Data for approximating transfer operator

The system has unstable equilibrium point at the origin and two stable equilibrium point at $(\pm 1, 0)$. The objective is to stabilize the unstable equilibrium point at the origin. We consider the

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state space $X = [-2, 2] \times [-2, 2]$. For the finite dimensional approximation we use 100 Gaussian radial basis function with $\sigma = 0.2$. The centers for the radial basis functions are chosen using K-mean clustering algorithm applied to data set generated for open loop system and as shown in Fig. 4.6. The control input u is quantized to $\mathcal{U} = [-4 : 0.5 : 4]$. In Figs. 4.7 and 4.8 we show the plots for the open loop and closed loop trajectories along with optimal cost and control inputs.

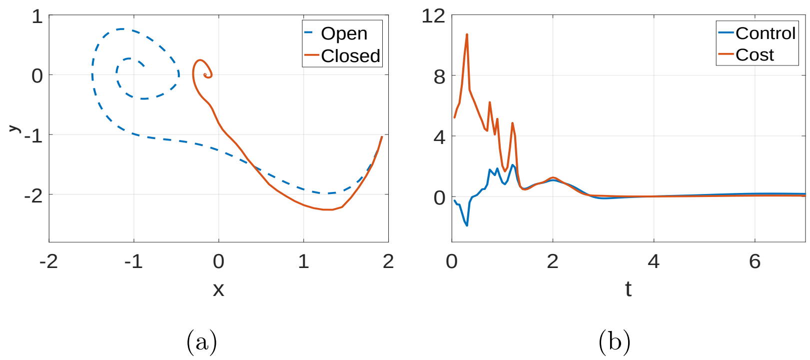

Figure 4.7: Duffing Oscillator, sample trajectory 1: a) Open loop and closed loop trajectory; b) Optimal cost and control values.

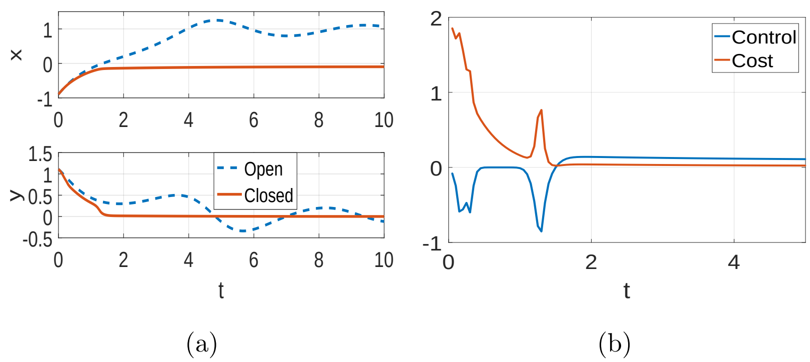

Figure 4.8: Duffing Oscillator, sample trajectory 2: a) Open loop and closed loop trajectory; b) Optimal cost and control values.

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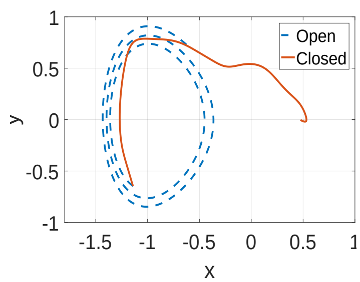
We observe that the open loop trajectories are attracted to the stable equilibrium at $(-1,0)$ or $(1,0)$ whereas using the designed control the closed loop trajectories converges to the equilibrium at origin.

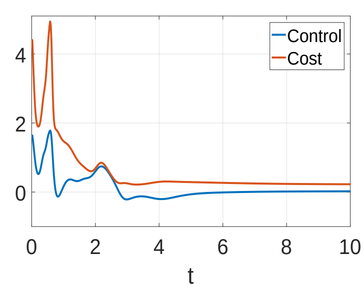
Basin Hopping in a Double Well

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1^3 + a x_1^2 + x_1 - a + u \end{aligned} \quad (4.6)$$

For parameter value of $a = 0.5$, the system has three equilibrium points at $(\pm 1, 0)$ and $(a, 0)$. The equilibrium points at $(\pm 1, 0)$ are stable and $(a, 0)$ is unstable. The objective is to stabilize the unstable equilibrium point at $(a, 0)$. Control quantization used for this example is $\mathcal{U} = [-2 : 0.2 : 2]$. For the finite dimension approximation, we construct 100 Gaussian radial basis functions with $\sigma = 0.22$. Using the designed control, the intended unstable equilibrium was successfully stabilized for almost all the initial conditions. In figures 4.9 and 4.10, we compare the open loop and close loop sample trajectories starting from two different initial conditions and corresponding optimal cost and control inputs.



(a)



(b)

Figure 4.9: Basin Hopping Double Well, sample trajectory 1: a) Open-loop and closed-loop trajectory; b) Optimal cost and control inputs.

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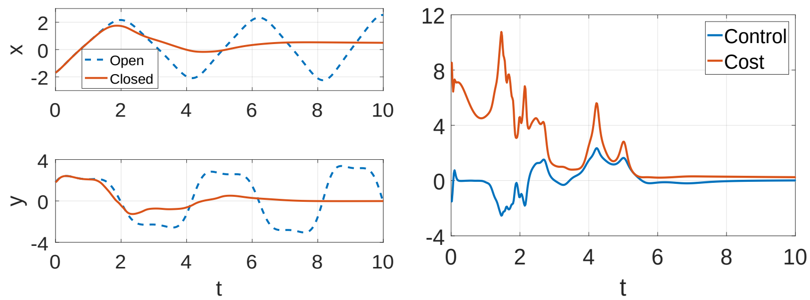

(a) (b)

Figure 4.10: Basin Hopping Double Well, sample trajectory 2: a) Open loop and closed loop trajectory; b) Optimal cost and control inputs.

Standard Map

$$
\begin{aligned}
x_{n+1} &= x_n + y_n + Ku \sin 2\pi x_n \pmod 1 \\
y_{n+1} &= y_n + Ku \sin 2\pi x_n
\end{aligned} \tag{4.7}
$$

Standard Map is one of the classical example of system exhibiting complex dynamics. The states of the standard map are canonical action-angle coordinates and they arise as a discretization of $1\frac{1}{2}$ degree of freedom Hamiltonian system. Control of standard maps are studied in Vaidya and Mezić (2004). For the uncontrolled standard map the entire state space $(x, y) \in [0, 1] \times [0, 1]$ is foliated with periodic and quasi periodic motion. The control objective is to stabilize the period 2 orbit located at $(0.25, 0.5)$ and $(0.75, 0.5)$. The parameter value of K is chosen to be equal to 0.25. Control is quantized to $\mathcal{U} = [-0.5 : 0.02 : 0.5]$.

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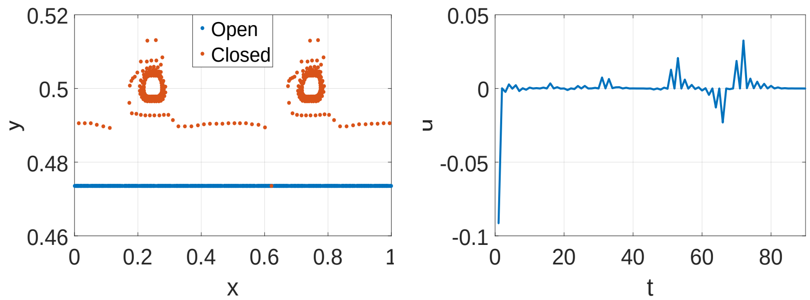
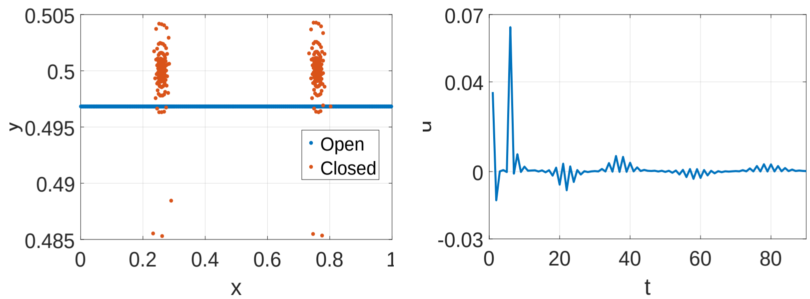


(a) Sample trajectory (b) Control values

Figure 4.11: Period 2-orbit stabilization for Standard Map: sample trajectory 1



(a) Sample trajectory (b) Control values

Figure 4.12: Standard Map, sample trajectory 2 and control

Open loop and closed loop control trajectories for the stabilization of period two orbit is shown in Fig. 4.11 and 4.12. For the sample initial point in Fig. 4.12a we observe excellent stabilization with corresponding optimal control values depicted in Fig. 4.12b.

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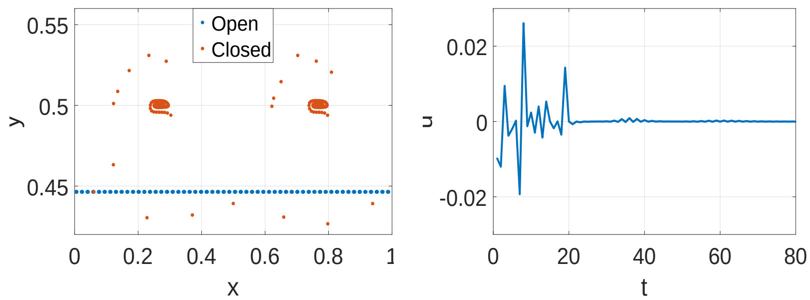
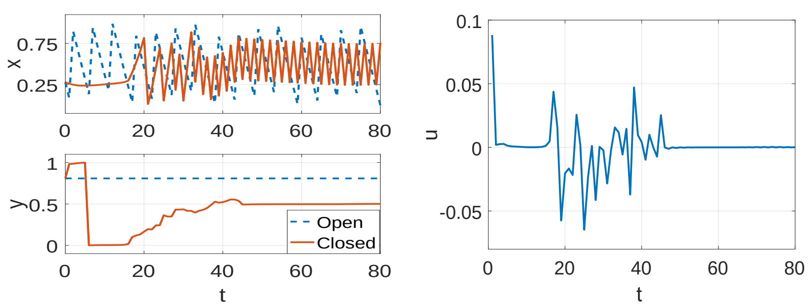


(a) Sample trajectory (b) Control values

Figure 4.13: Standard Map, sample trajectory 3 and control

In Fig. 4.14a,we observe that along the x direction the system trajectory toggle between two points $x = 0.25$ and $x = 0.75$ and along y axis the trajectory stabilize to $y = 0.5$. This is exactly what we ask for and thereby standard map period 2 orbits were stabilized perfectly.



(a) Sample trajectory (b) Control values

Figure 4.14: Standard Map, sample trajectory 4 and control

On a simulation of 5000 randomly chosen initial conditions, we chose 200 Gaussian radial basis functions for finite dimensional approximation, with $\sigma = 0.02$. This led approximately 93% of the trajectories to stabilization (captured within threshold distance of 0.05 from our intended period-2 orbits).

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CHAPTER 5. OPTIMAL STABILIZATION OF STOCHASTIC SYSTEMS FROM KNOWN DYNAMICS

A paper accepted by *American Control Conference 2017*

Apurba Kumar Das, Arvind Raghunathan, Umesh Vaidya

5.1 Abstract

In this paper, we develop linear transfer Perron-Frobenius operator-based approach for optimal stabilization of stochastic nonlinear systems. One of the main highlights of the proposed transfer operator based approach is that both the theory and computational framework developed for the optimal stabilization of deterministic dynamical systems in Raghunathan and Vaidya (2014) carries over to the stochastic case with little change. The optimal stabilization problem is formulated as an infinite-dimensional linear program. Set oriented numerical methods are proposed for the finite-dimensional approximation of the transfer operator and the controller. Simulation results are presented to verify the developed framework.

5.2 Introduction

Transfer operator-based methods have attracted lot of attention lately for problems involving dynamical systems analysis and design. In particular, transfer operator-based methods are used for identifying steady state dynamics of the system from the invariant measure of transfer operator, identifying almost invariant sets, and coherent structures Dellnitz and Junge (2002); Froyland and Dellnitz (2003); Froyland and Padberg (2009). The spectral analysis of transfer operators are also applied for reduced order modeling of dynamical systems with applications to building systems, power grid, and fluid mechanics Mehta and Vaidya (2005); Budišić et al. (2012). Operator-theoretic methods have also been successfully applied to address design problems in control dynamical sys-

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tems. In particular, transfer operator methods are used for almost everywhere stability verification, control design, nonlinear estimation, and for solving optimal sensor placement problem Surana and Banaszuk (2016); Raghunathan and Vaidya (2014); Vaidya and Mehta (2008); Rajaram et al. (2010); Vaidya et al. (2010); Vaidya (2007b); Sinha et al. (2016).

In this paper, we continue with the long series of work on the application of transfer operator methods for stability verification and stabilization of nonlinear systems. We develop an analytical and computational framework for the application of transfer operator methods for the stabilization of stochastic nonlinear systems. In Vaidya (2015), we introduced Lyapunov measure for stability verification of stochastic nonlinear systems. We proved that the existence of the Lyapunov measure verifies weaker set-theoretic notion of almost everywhere stochastic stability for discrete-time stochastic systems. Weaker notion of almost everywhere stability was introduced in Rantzer (2001) for continuous time deterministic systems and in Van Handel (2006) for continuous time stochastic systems. In this paper we extend the application of Lyapunov measure for optimal stabilization of stochastic nonlinear systems. Optimal stabilization of stochastic systems is posed as an infinite dimensional linear program. Set-oriented numerical methods are used for the finite dimensional approximation of the transfer operator and the linear program. A key advantage of the proposed transfer operator-based approach for stochastic stability analysis and controller synthesis is that all the stability results along with the computation framework carries over from the deterministic systems Surana and Banaszuk (2016); Raghunathan and Vaidya (2014) to the stochastic systems. The only difference in the stochastic setting is that the transfer Perron-Frobenius operator is defined for the stochastic system.

The results developed in this paper draw parallels from following papers. Lasserre, Hernndez-Lerma, and co-workers Hernández-Lerma and Lasserre (1996, 1998) formulated the control of Markov processes as a solution of the HJB equation. In Hernandez-Hernandez et al. (1996); Lasserre et al. (2005); Gaitsgory and Rossomakhine (2006), solutions to stochastic and deterministic optimal control problems are proposed, using a linear programming approach or using a sequence of LMI relaxations. Our paper also draws some connection to research on optimization and stabilization

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of controlled Markov chains discussed in Meyn (1999). Computational techniques based on the viscosity solution of the HJB equation is proposed for the approximation of the value function and optimal controls in Bardi and Capuzzo-Dolcetta (1997) [Chapter VI].

Our proposed method, in particular the computational approach, draws some similarity with the above discussed references on the approximation of the solution of the HJB equation Crespo and Sun (2000); Junge and Osinga (2004a); Bardi and Capuzzo-Dolcetta (1997); Grüne and Junge (2005). Our method, too, relies on discretization of state space to obtain globally optimal stabilizing control. However, our proposed approach differs from the above references in the following two fundamental ways. The first main difference arises due to adoption of non-classical weaker set-theoretic notion of almost everywhere stability for optimal stabilization. The second main difference compared to references Meyn (1999) and Bardi and Capuzzo-Dolcetta (1997) is in the use of the discount factor $\gamma > 1$ in the cost function. The discount factor plays an important role in controlling the effect of finite dimensional discretization or the approximation process on the true solution. In particular, by allowing for the discount factor γ to be greater than one, it is possible to ensure that the control obtained using the finite dimensional approximation is truly stabilizing for the nonlinear system Rajaram et al. (2010); Vaidya (2007a).

The paper is organized as follows. In section 2 we present brief overview of results from Vaidya (2015) on Lyapunov measure for stochastic stabilization. In section 5.3 results on application of Lyapunov measure for optimal stabilization are presented. In section 5.5, computational framework based on set-oriented numerical methods for finite dimensional approximation of Lyapunov measure and optimal control is presented. Simulation results are presented in section 5.6 followed by conclusions in section 5.7.

5.3 Lyapunov Measure for Stochastic Stability Analysis

Consider the discrete-time stochastic dynamical system,

$$x_{n+1} = T(x_n, \xi_n), \tag{5.1}$$

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where $x_n \in X \subset \mathbb{R}^d$ is a compact set. The random vectors, $\xi_0, \xi_1, \ldots$, are assumed independent identically distributed (i.i.d) and takes values in W with the following probability distribution,

$$\text{Prob}(\xi_n \in B) = v(B), \;\; \forall n, \;\; B \subset W,$$

and is the same for all n and v is the probability measure. The system mapping $T(x, \xi)$ is assumed continuous in x and for every fixed $x \in X$, it is measurable in ξ. The initial condition, x_0, and the sequence of random vectors, $\xi_0, \xi_1, \ldots$, are assumed independent. The basic object of study in our proposed approach to stochastic stability is a linear transfer, the Perron-Frobenius operator, defined as follows:

Definition 13 (Perron-Frobenius (P-F) operator) *Let $\mathcal{M}(X)$ be the space of finite measures on X. The Perron-Frobenius operator, $\mathbb{P} : \mathcal{M}(X) \to \mathcal{M}(X)$, corresponding to the stochastic dynamical system (5.1) is given by*

$$\begin{aligned}
[\mathbb{P}_T \mu](A) &= \int_X \left\{ \int_W \chi_A(T(x,y)) dv(y) \right\} d\mu(x) \\
&= E_\xi \left[\mu(T_\xi^{-1}(A)) \right], \qquad (5.2)
\end{aligned}$$

for $\mu \in \mathcal{M}(X)$, and $A \in \mathcal{B}(X)$, where $\chi_A(x)$ is an indicator function of set A.

Assumption 14 *We assume $x = 0$ is an equilibrium point of system (5.1), i.e., $T(0, \xi_n) = 0, \quad \forall n$, for any given sequence of random vectors $\{\xi_n\}$ taking values in set W.*

Assumption 15 (Local Stability) *We assume the trivial solution, $x = 0$, is locally stochastic, asymptotically stable. In particular, we assume there exists a neighborhood $\mathcal{O}$ of $x = 0$, such that for all $x_0 \in \mathcal{O}$,*

$$\text{Prob}\{T^n(x_0, \xi_0^n) \in \mathcal{O}\} = 1, \;\; \forall n \geq 0,$$

and

$$\text{Prob}\{\lim_{n \to \infty} T^n(x_0, \xi_0^n) = 0\} = 1.$$

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Assumption 14 is used in the decomposition of the P-F operator in section (5.3.1) and Assumption 15 is used in the proof of Theorem (20). In the following, we will use the notation $U(\epsilon)$ to denote the ϵ neighborhood of the origin for any positive value of $\epsilon > 0$. We have $0 \in U(\epsilon) \subset \mathcal{O}$.

We introduce the following definitions for stability of stochastic dynamical systems (5.1).

Definition 16 (a.e. stochastic stable with geometric decay) *For any given $\epsilon > 0$, let $U(\epsilon)$ be the ϵ neighborhood of the equilibrium point, $x = 0$. The equilibrium point, $x = 0$, is said to be almost everywhere, almost sure stable with geometric decay with respect to finite measure, $m \in \mathcal{M}(X)$, if there exists $0 < \alpha(\epsilon) < 1$, $0 < \beta < 1$, and $K(\epsilon) < \infty$, such that*

$$m\{x \in X : Prob\{T^n(x, \xi_0^n) \in B\} \geq \alpha^n\} \leq K\beta^n,$$

for all sets $B \in \mathcal{B}(X \setminus U(\epsilon))$, such that $m(B) > 0$.

We introduce the following definition of absolutely continuous and equivalent measures.

Definition 17 (Absolutely continuous measure) *A measure μ is absolutely continuous with respect to another measure, ϑ denoted as $\mu \prec \vartheta$, if $\mu(B) = 0$ for all $B \in \mathcal{B}(X)$ with $\vartheta(B) = 0$.*

Definition 18 (Equivalent measure) *The two measures, μ and ϑ, are equivalent ($\mu \approx \vartheta$) provided $\mu(B) = 0$, if and only if $\vartheta(B) = 0$ for $B \in \mathcal{B}(X)$.*

5.3.1 Decomposition of the P-F operator

Let $E = \{0\}$. Hence, $E^c = X \setminus E$. We write $T : E \cup E^c \times W \to X$. For any set $B \in \mathcal{B}(E^c)$, we can write

$$\begin{aligned} [\mathbb{P}_T\mu](B) &= \int_X \int_W \chi_B(T(x,y))dv(y)d\mu(x) \\ &= \int_{E^c} \int_W \chi_B(T(x,y))dv(y)d\mu(x). \end{aligned} \tag{5.3}$$

This is because $T(x, \xi) \in B$ implies $x \notin E$. Since set E is invariant, we define the restriction of the P-F operator on the complement set E^c. Thus, we can define the restriction of the P-F operator

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on the measure space $\mathcal{M}(E^c)$ as follows:

$$[\mathbb{P}_T^1\mu](B) = \int_{E^c}\int_W \chi_B(T(x,y))dv(y)d\mu(x),$$

for any set $B \in \mathcal{B}(E^c)$ and $\mu \in \mathcal{M}(E^c)$.

Next, the restriction $T : E \times W \to E$ can also be used to define a P-F operator denoted by

$$[\mathbb{P}_T^0\mu](B) = \int_B \chi_B(T(x,y))dv(y)d\mu(x),$$

where $\mu \in \mathcal{M}(E)$ and $B \subset \mathcal{B}(E)$.

The above considerations suggest a representation of the P-F operator, $\mathbb{P}$, in terms of $\mathbb{P}_0$ and $\mathbb{P}_1$. Indeed, this is the case, if one considers a splitting of the measured space,

$$\mathcal{M}(X) = \mathcal{M}_0 \oplus \mathcal{M}_1, \tag{5.4}$$

where $\mathcal{M}_0 := \mathcal{M}(E)$, $\mathcal{M}_1 := \mathcal{M}(E^c)$, and $\oplus$ stands for direct sum.

Then it follows the splitting defined by Eq. (5.4), the P-F operator has a lower-triangular matrix representation given by

$$\mathbb{P}_T = \begin{bmatrix} \mathbb{P}_T^0 & 0 \\ \times & \mathbb{P}_T^1 \end{bmatrix}. \tag{5.5}$$

Let $\xi_0^n = \{\xi_0, \dots, \xi_n\} \in \underbrace{W \times \ldots \times W}_{n} =: W^n$ and $F(x, \xi_0^n) := T^n(x, \xi_0^n) : X \times W^n \to X$ be the notation for the n times composition of the map $T : X \times W \to X$. Then, it is easy to show that the Perron-Frobenius operator, $\mathbb{P}_F : \mathcal{M}(X) \to \mathcal{M}(X)$, corresponding to system mapping F is given by

$$\mathbb{P}_F = \underbrace{\mathbb{P}_T\mathbb{P}_T \ldots \mathbb{P}_T}_{n} =: \mathbb{P}_T^n.$$

Using the lower triangular structure of the P-F operator in Eq. (5.5), one can write the $\mathbb{P}_T^n$ as follows:

$$\mathbb{P}_T^n = \begin{bmatrix} [\mathbb{P}_T^0]^n & 0 \\ \times & [\mathbb{P}_T^1]^n \end{bmatrix}. \tag{5.6}$$

Following is the definition of Lyapunov measure introduced for almost everywhere stability verification of a stochastic system.

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Definition 19 (Lyapunov measure) *A Lyapunov measure, $\bar{\mu} \in \mathcal{M}(X \setminus U(\epsilon))$, is defined as any positive measure finite outside the ϵ neighborhood of equilibrium point and satisfies*

$$[\mathbb{P}_T^1 \bar{\mu}](B) < \gamma \bar{\mu}(B) \tag{5.7}$$

for $0 < \gamma \leq 1$ and for all sets $B \in \mathcal{B}(X \setminus U(\epsilon))$.

The following theorem from Vaidya (2007a) provides the condition for a.e. stability with geometric decay.

Theorem 20 *An attractor set $\mathcal{A}$ for the stochastic dynamical system (5.1) is a.e. stochastic stable with geometric decay (Definition 16) with respect to finite measure m, if and only if for all $\epsilon > 0$ there exists a non-negative measure $\bar{\mu}$ which is finite on $\mathcal{B}(X \setminus U(\epsilon))$ and satisfies*

$$\gamma[\mathbb{P}_T^1 \bar{\mu}](B) - \bar{\mu}(B) = -m(B) \tag{5.8}$$

for all measurable sets $B \subset X \setminus U(\epsilon)$ and for some $\gamma > 1$.

Proof 21 *Proof for this theorem follows similar as its deterministic counterpart Raghunathan and Vaidya (2014).*

5.4 Lyapunov Measure for Optimal Stabilization

We consider the stabilization of stochastic dynamical systems of the form

$$x_{n+1} = T(x_n, u_n, \xi_n) = T_{\xi_n}(x_n, u_n)$$

where $x_n \in X \subset \mathbb{R}^q$ are the state, $u_n \in U \subset \mathbb{R}^d$ is the control input, and $\xi_n \in W \subset \mathbb{R}^p$ is random variable. The sequence of random variable $\xi_0, \xi_1, \ldots$ are assumed to independent identically distributed (i.i.d.) with following statistics

$$Prob\{\xi_n \in B\} = v(B), \quad \forall n, \quad B \subset W$$

where v is the probability measure. For each fixed value of ξ the mapping $T_\xi : X \times U \to X$ is assumed to be continuous in x and u and for every fixed values of x and u it is measurable in ξ.

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Both X and U are assumed compact. The objective is to design a deterministic feedback controller, $u_n = K(x_n)$, to optimally stabilize the attractor set $\mathcal{A}$.

We define the feedback control mapping $C : X \to Y := X \times U$ as $C(x) = (x, K(x))$. We denote by $\mathcal{B}(Y)$ the Borel-σ algebra on Y and $\mathcal{M}(Y)$ the vector space of real valued measures on $\mathcal{B}(Y)$. For any $\mu \in \mathcal{M}(X)$, the control mapping C can be used to define a measure, $\theta \in \mathcal{M}(Y)$, as follows:

$$\theta(D) := [\mathbb{P}_C \mu](D) = \mu(C^{-1}(D))$$
$$[\mathbb{P}_{C^{-1}}\theta](B) := \mu(B) = \theta(C(B)), \tag{5.9}$$

for all sets $D \in \mathcal{B}(Y)$ and $B \in \mathcal{B}(X)$. Since C is an injective function with θ satisfying (5.9), it follows from the theorem on disintegration of measure Furstenberg (1981) (Theorem 5.8) there exists a unique disintegration θ_x of the measure θ for μ almost all $x \in X$, such that $\int_Y f(y) d\theta(y) = \int_X \int_{C(x)} f(y) d\theta_x(y) d\mu(x)$, for any Borel-measurable function $f : Y \to \mathbb{R}$. In particular, for $f(y) = \chi_D(y)$, the indicator function for the set D, we obtain $\theta(D) = \int_X \int_{C(x)} \chi_D(y) d\theta_x(y) d\mu(x) = [\mathbb{P}_C \mu](D)$. Using the definition of the feedback controller mapping C, we write the feedback control system as

$$x_{n+1} = T(x_n, K(x_n), \xi_n) = T(C(x_n), \xi_n) =: T_{\xi_n} \circ C(x_n)$$

The system mapping $T : Y \times W \to X$ can be associated with P-F operators $\mathbb{P}_T : \mathcal{M}(Y) \to \mathcal{M}(X)$ as

$$[\mathbb{P}_T \theta](B) = \int_Y \left\{ \int_W \chi_B(T(y, \xi)) dv(\xi) \right\} d\theta(y).$$

For the feedback control system $T_\xi \circ C : X \times W \to X$, the P-F operator can be written as a product of $\mathbb{P}_{T_\xi}$ and $\mathbb{P}_C$. In particular, we obtain Raghunathan and Vaidya (2012)

$$[\mathbb{P}_{T_\xi \circ C} \mu](B) = \int_Y \left\{ \int_W \chi_B(T(y, \xi)) dv(\xi) \right\} d[\mathbb{P}_C \mu](y)$$
$$= \quad [\mathbb{P}_T \mathbb{P}_C \mu](B) = \int_X \int_{C(x)} \left\{ \int_W \chi_B(T(y, \xi)) dv(\xi) \right\} d\theta_x(y) d\mu(x).$$

The P-F operators, $\mathbb{P}_T$ and $\mathbb{P}_{\mathbb{C}}$, are used to define their restriction, $\mathbb{P}^1_T : \mathcal{M}(\mathcal{A}^c \times U) \to \mathcal{M}(\mathcal{A}^c)$, and $\mathbb{P}^1_C : \mathcal{M}(\mathcal{A}^c) \to \mathcal{M}(\mathcal{A}^c \times U)$ to the complement of the attractor set, respectively, in a way similar to Eq. (5.3.1).

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We define $X_1 := X \setminus U(\epsilon)$.

Assumption 22 *We assume there exists a feedback controller mapping $C_0(x) = (x, K_0(x))$, which locally stabilizes the invariant set $\mathcal{A}$, i.e., there exists a neighborhood V of $\mathcal{A}$ such that $T \circ C_0(V) \subset V$ and $x_n \to \mathcal{A}$ for all $x_0 \in V$; moreover $\mathcal{A} \subset U(\epsilon) \subset V$.*

Our objective is to construct the optimal stabilizing controller for almost every initial condition starting from X_1. Let $C_1 : X_1 \to Y$ be the stabilizing control map for X_1. The control mapping $C : X \to X \times U$ can be written as follows:

$$C(x) = \begin{cases} C_0(x) = (x, K_0(x)) & \text{for} \quad x \in U(\epsilon) \\ C_1(x) = (x, K_1(x)) & \text{for} \quad x \in X_1. \end{cases} \tag{5.10}$$

Furthermore, we assume the feedback control system $T_\xi \circ C : X \to X$ is non-singular with respect to the Lebesgue measure, m for fixed value of ξ. We seek to design the controller mapping, $C(x) = (x, K(x))$, such that the attractor set $\mathcal{A}$ is a.e. stable with geometric decay rate $\beta < 1$, while minimizing the following cost function,

$$\mathcal{C}_C(B) = \int_B \sum_{n=0}^{\infty} \gamma^n E_{\xi_0^n}[G(C(x_n), \xi_n)] dm(x), \tag{5.11}$$

where $x_0 = x$, the cost function $G : X \times U \times W \to \mathbb{R}$ is assumed a continuous non-negative real-valued function for each fixed value of ξ and is assumed to be measurable w.r.t. ξ for fixed value of x and u. Furthermore, $G(\mathcal{A}, 0, \xi) = 0$ for all ξ, $x_{n+1} = T_\xi \circ C(x_n)$, and $0 < \gamma < \frac{1}{\beta}$. The expectation $E_{\xi_0^n}$ in (5.11) denotes expectation over the sequence of random variable $\{\xi_0, \xi_1, \ldots, \xi_n\}$ and using the i.i.d. property of the sequence of random variable are taken with respect to product probability measure $\underbrace{dv \ldots, dv}_{n+1}$. Note, that in the cost function (5.11), γ is allowed greater than one and this is one of the main departures from the conventional optimal control problem, where $\gamma \leq 1$. However, under the assumption that the controller mapping C renders the attractor set a.e. stable with a geometric decay rate, $\beta < \frac{1}{\gamma}$, the cost function (5.11) is finite.

Remark 23 *To simplify the notation, in the following we will use the notion of the scalar product between continuous function $h \in \mathcal{C}^0(X)$ and measure $\mu \in \mathcal{M}(X)$ as $\langle h, d\mu \rangle_X := \int_X h(x) d\mu(x)$ Lasota and Mackey (1994).*

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The following theorem proves the cost of stabilization of the set $\mathcal{A}$ as given in Eq. (5.11) can be expressed using the control Lyapunov measure equation.

Theorem 24 *Let the controller mapping, $C(x) = (x, K(x))$, be such that the attractor set $\mathcal{A}$ for the feedback control system $T_\xi \circ C : X \to X$ is a.e. stable with geometric decay rate $\beta < 1$. Then, the cost function (5.11) is well defined for $\gamma < \frac{1}{\beta}$ and, furthermore, the cost of stabilization of the attractor set $\mathcal{A}$ with respect to Lebesgue almost every initial condition starting from set $B \in \mathcal{B}(X_1)$ can be expressed as follows:*

$$\begin{aligned}\mathcal{C}_C(B) &= \int_B \sum_{n=0}^{\infty} \gamma^n E_{\xi_0^n}[G(C(x_n), \xi_n)]dm(x) \\ &= \int_{\mathcal{A}^c \times U \times W} G(y, \xi) d[\mathbb{P}_C^1 \bar{\mu}_B](y) dv(\xi) \\ &= \left\langle G, d\mathbb{P}_C^1 \bar{\mu}_B dv \right\rangle_{\mathcal{A}^c \times U \times W}, \end{aligned} \tag{5.12}$$

where, $x_0 = x$ and $\bar{\mu}_B$ is the solution of the following control Lyapunov measure equation,

$$\gamma[\mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}_B](D) - \bar{\mu}_B(D) = -m_B(D), \tag{5.13}$$

for all $D \in \mathcal{B}(X_1)$ and where $m_B(\cdot) := m(B \cap \cdot)$ is a finite measure supported on the set $B \in \mathcal{B}(X_1)$.

Proof 25

By appropriately selecting the measure on the right-hand side of the control Lyapunov measure equation (5.13) (i.e., m_B), stabilization of the attractor set with respect to a.e. initial conditions starting from a particular set can be studied. The minimum cost of stabilization is defined as the minimum over all a.e. stabilizing controller mappings, C, with a geometric decay as follows:

$$\mathcal{C}^*(B) = \min_C \mathcal{C}_C(B). \tag{5.14}$$

Next, we write the infinite dimensional linear program for the optimal stabilization of the attractor set $\mathcal{A}$. Towards this goal, we first define the projection map, $P_1 : \mathcal{A}^c \times U \to \mathcal{A}^c$ as: $P_1(x, u) = x$, and

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denote the P-F operator corresponding to P_1 as $\mathbb{P}_{P_1} : \mathcal{M}(\mathcal{A}^c \times U) \to \mathcal{M}(\mathcal{A}^c)$, which can be written as $[\mathbb{P}^1_{P_1}\theta](D) = \int_{\mathcal{A}^c \times U} \chi_D(P_1(y))d\theta(y) = \int_{D\times U} d\theta(y) = \mu(D)$. Using this definition of projection mapping, P_1, and the corresponding P-F operator, we can write the linear program for the optimal stabilization of set B with unknown variable θ as follows:

$$\min_{\theta\geq 0} \langle G, d\theta dv\rangle_{\mathcal{A}^c\times U\times W}, \text{ s.t. } \gamma[\mathbb{P}^1_T\theta](D) - [\mathbb{P}^1_{P_1}\theta](D) = -m_B(D), \tag{5.15}$$

for $D \in \mathcal{B}(X_1)$.

Remark 26 *Observe the geometric decay parameter satisfies $\gamma > 1$. This is in contrast to most optimization problems studied in the context of Markov-controlled processes, such as in Lasserre and Hernández-Lerma Hernández-Lerma and Lasserre (1996). Average cost and discounted cost optimality problems are considered in Hernández-Lerma and Lasserre (1996); Bardi and Capuzzo-Dolcetta (1997). The additional flexibility provided by $\gamma > 1$ guarantees the controller obtained from the finite dimensional approximation of the infinite dimensional program (5.15) also stabilizes the attractor set for system (5.4). For a more detailed discussion on the role of γ on the finite dimensional approximation, we refer readers to the online version of the paper Raghunathan and Vaidya (2012).*

5.5 Computational Approach

We discretize the state-space and control space for the purposes of computations as described below. Borrowing the notation from Vaidya et al. (2010), let $\mathcal{X}_N := \{D_1, ..., D_i, ..., D_N\}$ denote a finite partition of the state-space $X \subset \mathbb{R}^q$. The measure space associated with $\mathcal{X}_N$ is $\mathbb{R}^N$. We assume without loss of generality that the attractor set, $\mathcal{A}$, is contained in D_N, that is, $\mathcal{A} \subseteq D_N$. The control space, U, is quantized and the control input is assumed to take only finitely many control values from the quantized set, $\mathcal{U}_M = \{u^1, \dots, u^a, \dots, u^M\}$, where $u^a \in \mathbb{R}^d$. The partition, $\mathcal{U}_M$, is identified with the vector space, $\mathbb{R}^{d\times M}$. Similarly, the space of uncertainty, W, and the probability measure v is quantized and are assumed to take only finitely many values $\mathcal{W} = \{\xi^1, \dots, \xi^\ell, \dots, \xi^L\}$, and $\vartheta = \{v^1, \dots, v^\ell, \dots, v^L\}$ where $\xi^\ell \in \mathbb{R}^p$ and $0 \leq v^\ell \leq 1$ for all ℓ and $\sum_{\ell=1}^L v^\ell = 1$. The discrete

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probability measure on the finite dimensional uncertainty space is assigned as follows:

$$Prob\{\xi_n = \xi^\ell\} = v^\ell, \quad \forall n, \quad \ell = 1, \ldots, L.$$

The space of uncertainty is identified with finite dimensional space $\mathbb{R}^{p\times L}$. The system map that results from choosing the controls $u = u^a$ and uncertainty value $\xi = \xi^\ell$ is denoted by T_{u^a,ξ^ℓ} and the corresponding P-F operator is denoted as $P_{T_{u^a,\xi^\ell}} \in \mathbb{R}^{N\times N}$. Note that for system mapping T_{u^a,ξ^ℓ}, the control on all sets of the partition is $u(D_i) = u^a$, for all $D_i \in \mathcal{X}_N$. For brevity of notation, we will denote the P-F matrix $P_{T_{u^a,\xi^\ell}}$ by $P_{T_{a,\ell}}$ and its entries are calculated as

$$(P_{T_{a,\ell}})_{(ij)} := \frac{m(T^{-1}_{u^a,\xi^\ell}(D_j) \cap D_i)}{m(D_i)},$$

where m is the Lebesgue measure and $(P_{T_{a,\ell}})_{(ij)}$ denotes the (i,j)-th entry of the matrix. Since $T_{u^a,\xi^\ell} : X \to X$, we have $P_{T_{a,\ell}}$ is a Markov matrix. Additionally, $P^1_{T_{a,\ell}} : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ will denote the finite dimensional counterpart of the P-F operator restricted to $\mathcal{X}_N \setminus D_N$, the complement of the attractor set. It is easily seen that $P^1_{T_{a,\ell}}$ consists of the first $(N-1)$ rows and columns of $P_{T_{a,\ell}}$.

In Vaidya and Mehta (2008) and Vaidya et al. (2010), stability analysis and stabilization of the attractor set are studied, using the above finite dimensional approximation of the P-F operator. The finite dimensional approximation of the P-F operator results in a weaker notion of stability, referred to as coarse stability Vaidya and Mehta (2008); Raghunathan and Vaidya (2012). Roughly speaking, coarse stability means stability modulo attractor sets with domain of attraction smaller than the size of cells within the partition.

With the above quantization of the control space and partition of the state space, the determination of the control $u(x) \in U$ (or equivalently $K(x)$) for all $x \in \mathcal{A}^c$ has now been cast as a problem of choosing $u_N(D_i) \in \mathcal{U}_M$ for all sets $D_i \subset \mathcal{X}_N$.

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The finite dimensional approximation of the optimal stabilization problem (5.15) is equivalent to solving the following finite-dimensional LP:

$$\min_{\theta^a,\mu\geq 0} \sum_{a=1}^{M}\left[\sum_{\ell=1}^{L} v^\ell (G^{a,\ell})'\right]\theta^a,$$
$$\text{s.t. } \gamma\sum_{a=1}^{M}\left[\sum_{\ell=1}^{L} v^\ell (P_{T_{a,\ell}})'\right]\theta^a - \sum_{a=1}^{M}\theta^a = -m, \tag{5.16}$$

where we have used the notation $(\cdot)'$ for the transpose operation, $m \in \mathbb{R}^{N-1}$ and $(m)_{(j)} > 0$ denote the support of Lebesgue measure, m, on the set D_j, $G^{a,\ell} \in \mathbb{R}^{N-1}$ is the cost defined on $\mathcal{X}_N \setminus D_N$ with $(G^{a,\ell})_{(j)}$ the cost associated with using control action u^a on set D_j with uncertainty value $\xi_n = \xi^\ell$; $\theta^a \in \mathbb{R}^{N-1}$ are, respectively, the discrete counter-parts of infinite-dimensional measure quantities in (5.15). We define following quantities

$$G^a := \sum_{\ell=1}^{L} v^\ell G^{a,\ell}, \quad P_{T_a} := \sum_{\ell=1}^{L} v^\ell P_{T_{a,\ell}}$$

to rewrite finite-dimensional LP (5.16) as follows:

$$\min_{\theta^a,\mu\geq 0} \sum_{a=1}^{M}(G^a)'\theta^a, \quad \text{s.t. } \gamma\sum_{a=1}^{M}(P_{T_a})'\theta^a - \sum_{a=1}^{M}\theta^a = -m, \tag{5.17}$$

In the LP (5.17), we have not enforced the constraint,

$$(\theta^a)_{(j)} > 0 \text{ for exactly one } a \in \{1, ..., M\}, \tag{5.18}$$

for each $j = 1, ..., (N-1)$. The above constraint ensures the control on each set in unique. We prove in the following the uniqueness can be ensured without enforcing the constraint, provided the LP (5.17) has a solution. To this end, we introduce the dual LP associated with the LP in (5.17). The dual to the LP in (5.17) is,

$$\max_{V} \; m'V, \quad \text{s.t. } V \leq \gamma P^1_{T_a} V + G^a \;\forall a = 1, ..., M. \tag{5.19}$$

In the above LP (5.19), V is the dual variable to the equality constraints in (5.17).

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5.5.1 Existence of solutions to the finite LP

We make the following assumption throughout this section.

Assumption 27 *There exists* $\theta^a \in \mathbb{R}^{N-1}\ \forall\ a = 1, \ldots, M$*, such that the LP in* (5.17) *is feasible for some* $\gamma > 1$*.*

Note, Assumption 27 does not impose the requirement in (5.18). For the sake of simplicity and clarity of presentation, we will assume that the measure, m, in (5.17) is equivalent to the Lebesgue measure and $G > 0$. Satisfaction of Assumption 27 can be verified using the following algorithm.

Algorithm 1 1) Set $\mathcal{I} := 1, ..., N-1$, $\mathcal{I}_0 := N$, $L = 0$. 2) Set $\mathcal{I}_{L+1} := \emptyset$. 3) For each $i \in \mathcal{I} \setminus \{\mathcal{I}_0 \cup \ldots \cup \mathcal{I}_L\}$ do a) Pick the smallest $a \in 1, ..., M$ such that $(P_{T_a})_{(ij)} > 0$ for some $j \in \mathcal{I}_L$. b) If a exists then, set $u_N(D_i) := u^a$, $\mathcal{I}_{L+1} := \mathcal{I}_{L+1} \cup \{i\}$. 4) End For 5) If $\mathcal{I}_0 \cup \ldots \cup \mathcal{I}_L = \mathcal{I}$ then, set $L = L + 1$. STOP. 6) If $\mathcal{I}_{L+1} = \emptyset$ then, STOP. 7) Set $L = L + 1$. Go to Step 2.

The algorithm iteratively adds to $\mathcal{I}_{L+1}$, set D_i, which has a non-zero probability of transition to any of the sets in $\mathcal{I}_L$. In graph theory terms, the above algorithm iteratively builds a tree starting with the set $D_N \supseteq \mathcal{A}$. If the algorithm terminates in Step 6, then we have identified sets $\mathcal{I} \setminus \{\mathcal{I}_0 \cup \ldots \cup \mathcal{I}_L\}$ that cannot be stabilized with the controls in $\mathcal{U}_M$. If the algorithm terminates at Step 5, then we show in the Lemma below that a set of stabilizing controls exist.

Lemma 28 *Let* $\mathcal{X}_N = \{D_1, \ldots, D_n\}$ *be a partition of the state space,* X*, and* $\mathcal{U}_M = \{u^1, \ldots, u^M\}$ *be a quantization of the control space,* U*. Suppose Algorithm* 1 *terminates in Step* 5 *after* $L^{\max}$ *iterations, then the controls* u_N *identified by the algorithm renders the system coarse stable.*

Proof 29 *Let* $P_{T_{u_N}}$ *represent the closed loop transition matrix resulting from the controls identified by Algorithm 1. Suppose* $\mu \in \mathbb{R}^{N-1}$*,* $\mu \geq 0, \mu \neq 0$ *be any initial distribution supported on the complement of the attractor set* $\mathcal{X}_N \setminus D_N$*. By construction,* μ *has a non-zero probability of entering the attractor set after* $L^{\max}$ *transitions. Hence,*

$$\sum_{i=1}^{N-1} (\mu'(P^1_{T_{u_N}})^{L^{\max}})_{(i)} < \sum_{i=1}^{N-1} (\mu)_{(i)} \implies \lim_{n\to\infty} (P^1_{T_{u_N}})^{nL^{\max}} \longrightarrow 0.$$

Thus, the sub-Markov matrix $P^1_{T_{u_N}}$ *is transient and implies the claim.*

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Algorithm 1 is less expensive than the approaches proposed in Vaidya et al. (2010), where a mixed integer LP and a nonlinear programming approach were proposed. The strength of our algorithm is that it is guaranteed to find deterministic stabilizing controls, if they exist. The following lemma shows an optimal solution to (5.17) exists under Assumption 27.

Lemma 30 *Consider a partition $\mathcal{X}_N = \{D_1, \ldots, D_N\}$ of the state-space X with attractor set $\mathcal{A} \subseteq D_N$ and a quantization $\mathcal{U}_M = \{u^1, \ldots, u^M\}$ of the control space U. Suppose Assumption 27 holds for some $\gamma > 1$ and for $m, G > 0$. Then, there exists an optimal solution, θ, to the LP (5.17) and an optimal solution, V, to the dual LP (5.19) with equal objective values, ($\sum_{a=1}^{M} (G^a)' \theta^a = m'V$) and θ, V bounded.*

Proof 31 *From Assumption 27, the LP (5.17) is feasible. Observe the dual LP in (5.19) is always feasible with a choice of $V = 0$. The feasibility of primal and dual linear programs implies the claim as a result of LP strong duality Mangasarian (1994).*

Remark 32 *Note, existence of an optimal solution does not impose a positivity requirement on the cost function, G. In fact, even assigning $G = 0$ allows determination of a stabilizing control from the Lyapunov measure equation (5.17). In this case, any feasible solution to (5.17) suffices.*

The next result shows the LP (5.17) always admits an optimal solution satisfying (5.18).

Lemma 33 *Given a partition $\mathcal{X}_N = \{D_1, \ldots, D_N\}$ of the state-space, X, with attractor set, $\mathcal{A} \subseteq D_N$, and a quantization, $\mathcal{U}_M = \{u^1, \ldots, u^M\}$, of the control space, U. Suppose Assumption 27 holds for some $\gamma > 1$ and for $m, G > 0$. Then, there exists a solution $\theta \in \mathbb{R}^{N-1}$ solving (5.17) and $V \in \mathbb{R}^{N-1}$ solving (5.19) for any $\gamma \in [1, \overline{\gamma}_N)$. Further, the following hold at the solution: 1) For each $j = 1, ..., (N-1)$, there exists at least one $a_j \in 1, ..., M$, such that $(V)_{(j)} = \gamma (P^1_{T_{a_j}} V)_{(j)} + (G^{a_j})_{(j)}$ and $(\theta^{a_j})_{(j)} > 0$. 2) There exists a $\tilde{\theta}$ that solves (5.17), such that for each $j = 1, ..., (N-1)$, there is exactly one $a_j \in 1, ..., M$, such that $(\tilde{\theta}^{a_j})_{(j)} > 0$ and $(\tilde{\theta}^{a'})_{(j)} = 0$ for $a' \neq a_j$.*

Proof 34 *From the assumptions, we have that Lemma 30 holds. Hence, there exists $\theta \in \mathbb{R}^{N-1}$ solving (5.17) and $V \in \mathbb{R}^{N-1}$ solving (5.19) for any $\gamma \in [1, \overline{\gamma}_N)$. Further, θ and V satisfy following*

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the first-order optimality conditions Mangasarian (1994),

$$\begin{aligned} \sum_{a=1}^{M} \theta^a - \gamma \sum_{a=1}^{M} (P^1_{T_a})'\theta^a = m \\ V \leq \gamma P^1_{T_a} V + G^a \perp \theta^a \geq 0 \ \forall a = 1, ..., M. \end{aligned} \tag{5.20}$$

We will prove each of the claims in order.

Claim 1: *Suppose, there exists* $j \in 1, ..., (N-1)$*, such that* $(\theta^a)_{(j)} = 0$ *for all* $a = 1, ..., M$*. Substituting in the optimality conditions (5.20), one obtains,*

$$\gamma \sum_{a=1}^{M} ((P^1_{T_a})'\theta^a)_{(j)} = -(m)_{(j)}$$

which cannot hold, since, $P^1_{T_a}$ *has non-negative entries,* $\gamma > 0$ *and* $\theta^a \geq 0$*. Hence, there exists at least one* a_j *such that* $(\theta^{a_j})_{(j)} > 0$*. The complementarity condition in (5.20) then requires that* $(V)_{(j)} = (\gamma P^1_{T_{a_j}} V)_{(j)} + (G^{a_j})_{(j)}$*. This proves the first claim.*

Claim 2: *Denote* $a(j) = \min\{a | (\theta^a)_{(j)} > 0\}$ *for each* $j = 1, ..., (N-1)$*. The existence of* $a(j)$ *for each* j *follows from statement 1. Define* $P^1_{T_u} \in \mathbb{R}^{(N-1)\times(N-1)}$ *and* $G^u \in \mathbb{R}^{N-1}$ *as follows:*

$$\begin{aligned} (P^1_{T_u})_{(ji)} &:= (P^1_{T_{a(j)}})_{(ji)} \ \forall i = 1, ..., (N-1) \\ (G^u)_{(j)} &:= (G^{a(j)})_{(j)} \end{aligned} \tag{5.21}$$

for all $j = 1, ..., (N-1)$*. From the definition of* $P^1_{T_u}$*,* G^u *and the complementarity condition in (5.20), it is easily seen that* V *satisfies*

$$V = \gamma P^1_{T_u} V + G^u = \lim_{n\to\infty} ((\gamma P^1_{T_u})^n V + \sum_{k=0}^{n} (\gamma P^1_{T_u})^k G^u). \tag{5.22}$$

Since V *is bounded and* $G^u > 0$*, it follows that* $\rho(P^1_{T_u}) < 1/\gamma$*. Define* $\tilde{\theta}$ *as,*

$$\begin{bmatrix} (\tilde{\theta}^{a(1)})_{(1)} \\ \vdots \\ (\tilde{\theta}^{a(N-1)})_{(N-1)} \end{bmatrix} = (I_{N-1} - \gamma (P^1_{T_u})')^{-1} m \tag{5.23a}$$

$$(\tilde{\theta}^a)_{(j)} = 0 \ \forall j = 1, ..., (N-1), \quad a \neq a(j). \tag{5.23b}$$

The above is well-defined, since we have already shown that $\rho(P^1_{T_u}) < 1/\gamma$*.*

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From the construction of $\tilde{\theta}$*, we have that for each* j *there exists only one* a_j*, namely* $a(j)$*, for which* $(\tilde{\theta}^{a(j)})_{(j)} > 0$*. It remains to show that* $\tilde{\theta}$ *solves* (5.17)*. For this, observe*

$$\begin{aligned} \sum_{a=1}^{M} (G^a)' \tilde{\theta}^a &\stackrel{(5.23b)}{=} \sum_{j=1}^{N-1} (G_j^{a(j)} \tilde{\theta}^{a(j)})_{(j)} \stackrel{(5.23a)}{=} (G^u)' (I_{N-1} - \gamma (P^1_{T_u})')^{-1} m \\ &= ((I_{N-1} - \gamma P^1_{T_u})^{-1} G^u)' m \stackrel{(5.22)}{=} V' m. \end{aligned} \tag{5.24}$$

The primal and dual objectives are equal with the above definition of $\tilde{\theta}$*. Hence,* $\tilde{\theta}$ *solves* (5.17)*. The claim is proved.*

The following theorem states the main result.

Theorem 35 *Consider a partition* $\mathcal{X}_N = \{D_1, \ldots, D_N\}$ *of the state-space,* X*, with attractor set,* $\mathcal{A} \subseteq D_N$*, and a quantization,* $\mathcal{U}_M = \{u^1, \ldots, u^M\}$*, of the control space,* U*. Suppose Assumption 27 holds for some* $\gamma > 1$ *and for* $m, G > 0$*. Then, the following statements hold: 1) there exists a bounded* θ*, a solution to* (5.17) *and a bounded* V*, a solution to* (5.19)*; 2) the optimal control for each set,* $j = 1, ..., (N-1)$*, is given by* $u(D_j) = u^{a(j)}$*, where* $a(j) := \min\{a | (\theta^a)_{(j)} > 0\}$*; 3)* μ *satisfying* $\gamma (P^1_{T_u})' \mu - \mu = -m$ *, where* $(P^1_{T_u})_{(ji)} = (P^1_{T_{a(j)}})_{(ji)}$ *is the Lyapunov measure for the controlled system.*

Proof 36 *Assumption 27 ensures that the linear programs* (5.17) *and* (5.19) *have a finite optimal solution (Lemma* (30)*). This proves the first claim of the theorem and also allows the applicability of Lemma 33. The remaining claims follow as a consequence.*

Although the results in this section assumed the measure, m, is equivalent to Lebesgue, this can be easily relaxed to the case where m is absolutely continuous with respect to Lebesgue and is of interest where the system is not everywhere stabilizable. If it is known there are regions of the state-space not stabilizable, then m can be chosen such that its support is zero on these regions. If the regions are not known a priori then, (5.17) can be modified to minimize the l_1-norm of the constraint residuals. This is similar to the *feasibility phase* commonly employed in LP algorithms Wright (1997).

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5.6 Numerical Implementation

5.6.1 Stochastic damping in system

5.6.1.1 Inverted Pendulum on a cart:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{a\sin(x_1) - 0.5m_r x_2^2 \sin(2x_1) - b\cos(x_1)u}{1.33 - m_r \cos^2(x_1)} - 2\zeta\sqrt{a}x_2 \quad (5.25)$$

where $g = 9.8, l = 0.5, m = 2, M = 8, \zeta = 0, m_r = \frac{m}{(m+M)}, a = \frac{g}{l}, b = \frac{m_r}{ml}$. The cost function is assumed to be $G(x,u) = x_1^2 + x_2^2 + u^2$. The damping parameter ζ is assumed to be random and uniformly distributed with mean zero and uniformly supported on the interval $[-\sigma, \sigma]$. For uncontrolled system, $u = 0$, there are two equilibrium points, one equilibrium point at $(\pi, 0)$ is stable in Lyapunov sense with eigenvalues of linearization on the $j\omega$ axis, the second equilibrium point at the origin is a saddle and unstable. The objective is to optimally stabilize the saddle equilibrium point at the origin. For the purpose of discretization we use $\delta t = 0.1$ as time discretization for the simulations. The state space X is chosen to be limited in $[-\pi, \pi] \times [-10, 10]$ and is partitioned into $70 \times 70 = 4900$ boxes. For constructing the P-F matrix 10 initial conditions are located in each box. The control set is discretized as follows $\mathcal{U} = \{-80, -70 \ldots, -10, 0, 10, \ldots, 70, 80\}$. Similarly, the range of random parameter $[-\sigma, \sigma]$ is divided into 10 uniformly spaced discrete values for random parameter ξ.

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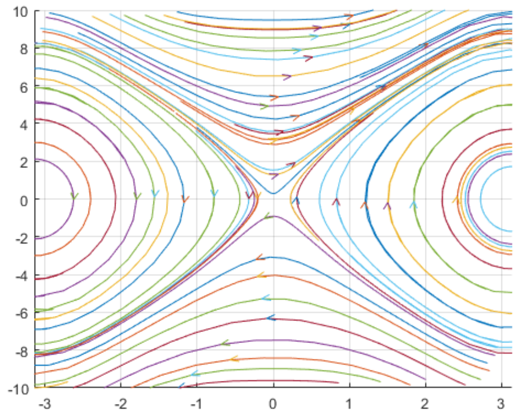
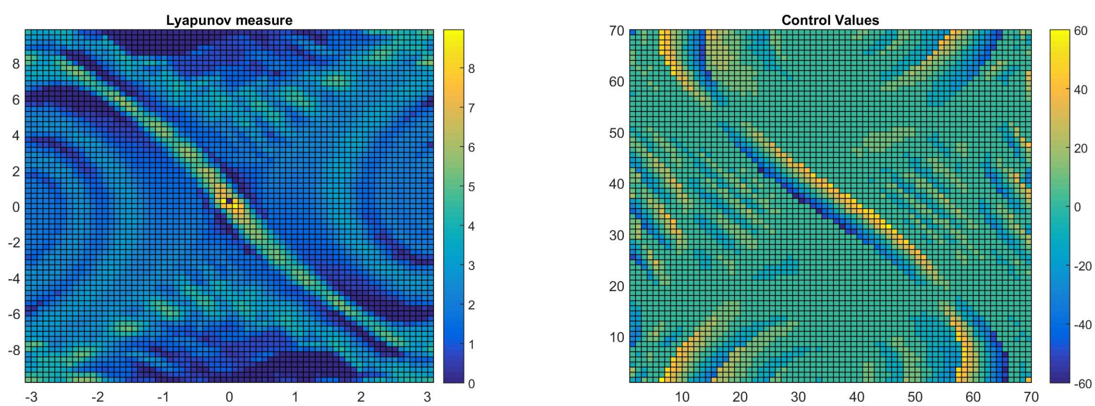


Figure 5.1: Phase portrait of inverted pendulum on a cart

In Fig. 5.1, we show the phase portrait of the uncontrolled inverted pendulum. The objective is to stabilize the saddle point at the origin.

Case 1: $\sigma = 0.02$



(a) (b)

Figure 5.2: Case 1: a) Lyapunov measure; b) Control measure

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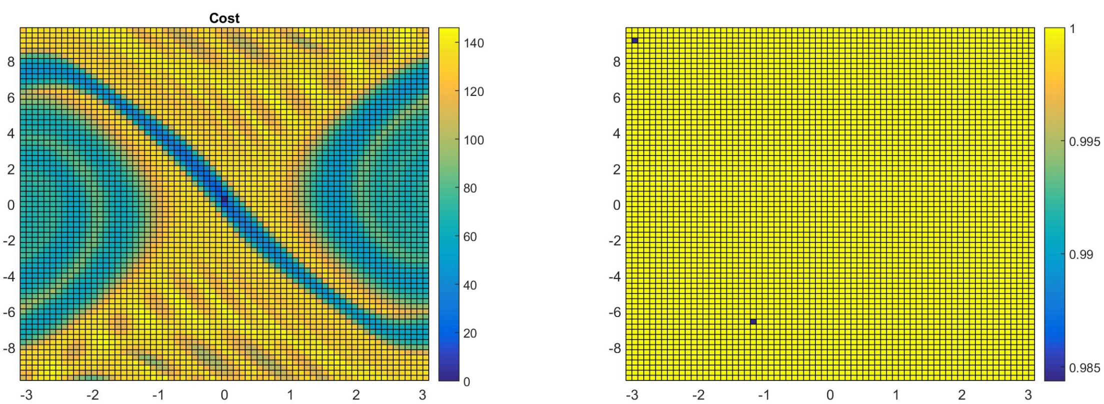
From figure 5.2, trajectories of the Lyapunov measure look similar to the phase portrait of the deterministic system. We observe the control values to stabilize the system as well.

Corresponding cost values in Fig. 5.3 show that the trajectories in the vertical middle section of phase plane including all of top and bottom required very high control cost to stabilize except the stable manifold directed from the top left to bottom right where there is very low or no cost of stabilization.



(a) (b)

Figure 5.3: Case 1: a) Optimal Cost; b) Success of stabilization

Now, to see the performance of stabilization we plot the fraction of points in each boxes which were stabilized to the attractor set at the origin. The system has been everywhere stabilized except two cells which have very small percentage of points likely to remain unstable. This can be occurring from approximation error in finite dimensions.

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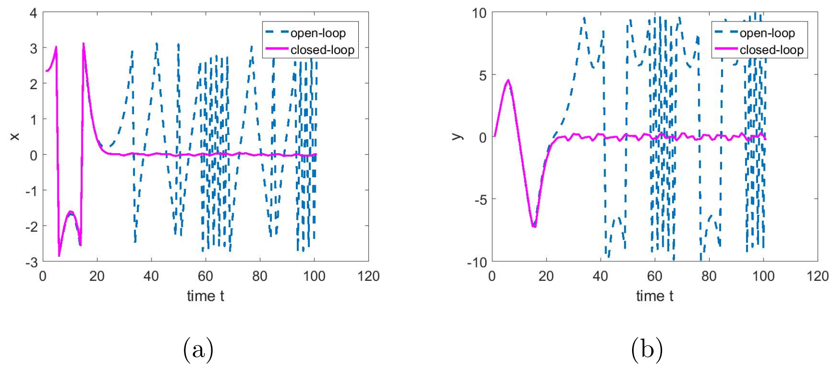

Figure 5.4: Case 1: a) x-trajectory; b) y-trajectory

In Fig. 5.4, we see how the closed loop trajectories were captured to the equilibrium.

Case 2: $\sigma = 0.1$

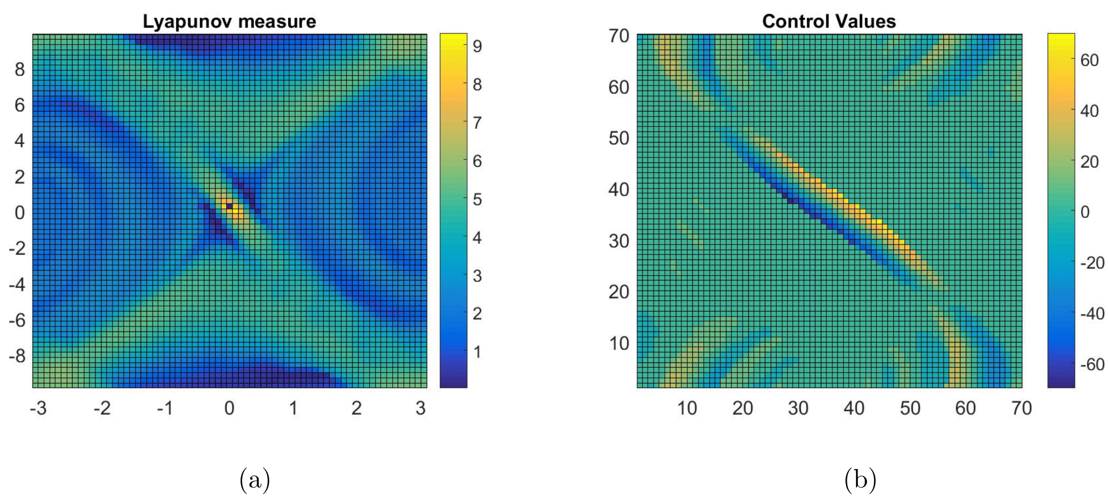

Figure 5.5: Case 2: a) Lyapunov measure; b) Control measure

With increase of stochasticity in ζ through parameter σ, we observe the performance of stabilization deteriorated significantly. Lyapunov measure is quite different from the last case. Behaviour found in phase portrait of the deterministic system is not identifiable in this case. For example, the stable manifold is hardly visible in Fig. 5.5a.

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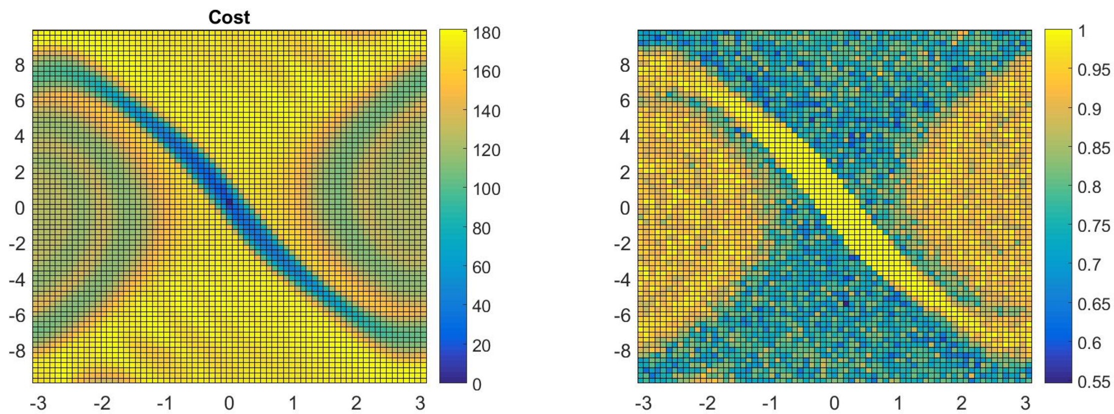
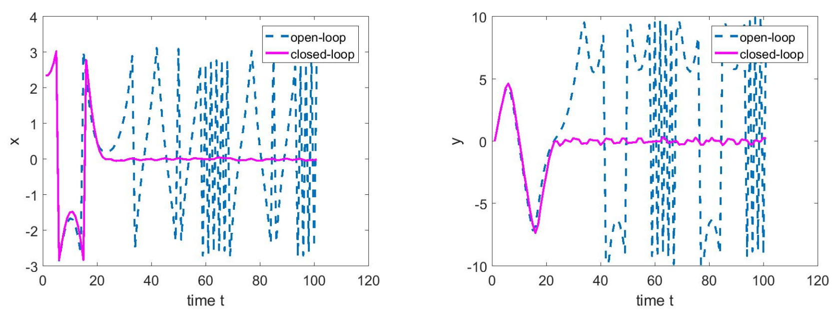


(a) (b)

Figure 5.6: Case 2: a) Cost of stabilization; b) Success of stabilization



(a) (b)

Figure 5.7: Case 2: a) x-trajectory; b) y-trajectory

In Fig. 5.7, we see how the closed loop trajectories were captured to the equilibrium even for high stochasticity. However, from Fig. 5.6b, we see many cells failed in stabilization for the available control set $\mathcal{U}$. This implies, we might need a set of higher control bound or denser control values for stabilizing this highly stochastic system.

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5.6.1.2 Vanderpol Oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \zeta(1 - x_1^2)x_2 - x_1 - u \tag{5.26}$$

The parameter ζ is assumed to be stochastic with mean 1 and uniformly distributed between interval $[1-\sigma, 1+\sigma]$. The system has a limit cycle as shown in Fig. 5.8 with unstable equilibrium point at the origin. The objective is to optimally stabilize the unstable equilibrium at the origin. The control set is discretized to $\mathcal{U} = \{-15, -14, -13..., 13, 14, 15\}$

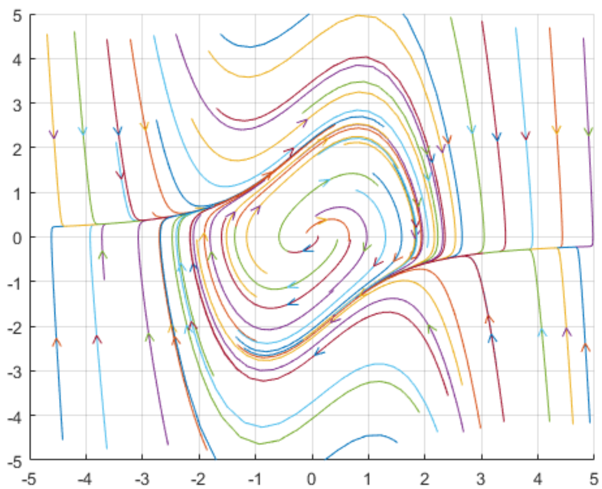

Figure 5.8: Phase portrait of Vanderpol Oscillator

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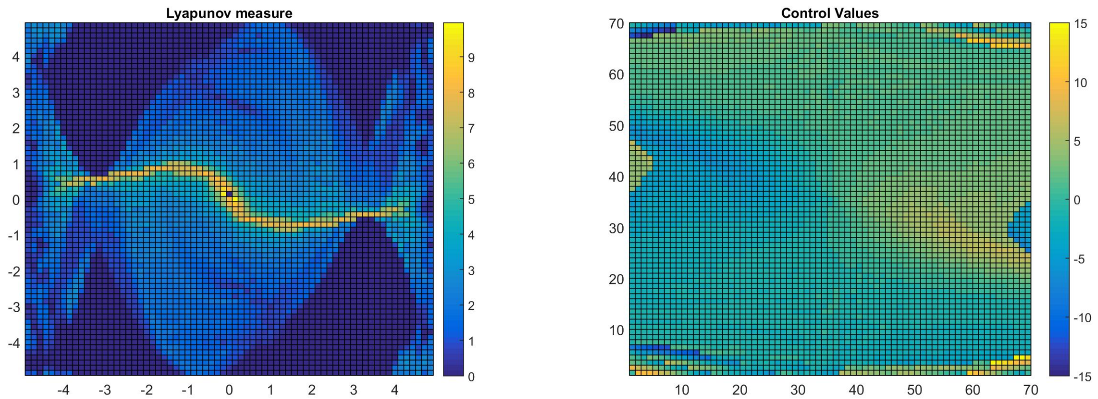
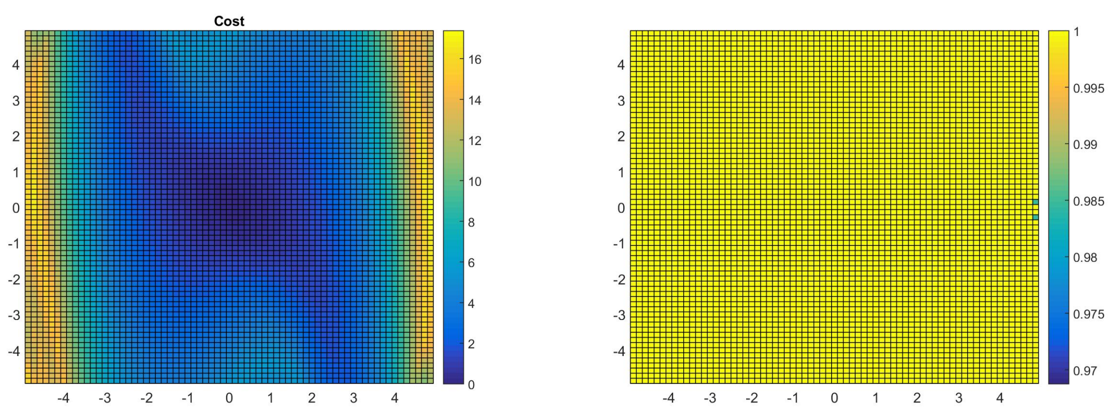
Case 1: $\sigma = 0.05$



(a) (b)

Figure 5.9: Case 1: a) Lyapunov measure; b) Control measure



(a) (b)

Figure 5.10: Case 1: a) Cost of stabilization; b) Percentage of points stabilized

Lyapunov measure in Fig. 5.9a does not fully reflect the phase portrait of the system since we are concerned about the invariant set at origin instead of the largely dominant limit cycle. Close system stabilizes all initial conditions as observed in 5.10

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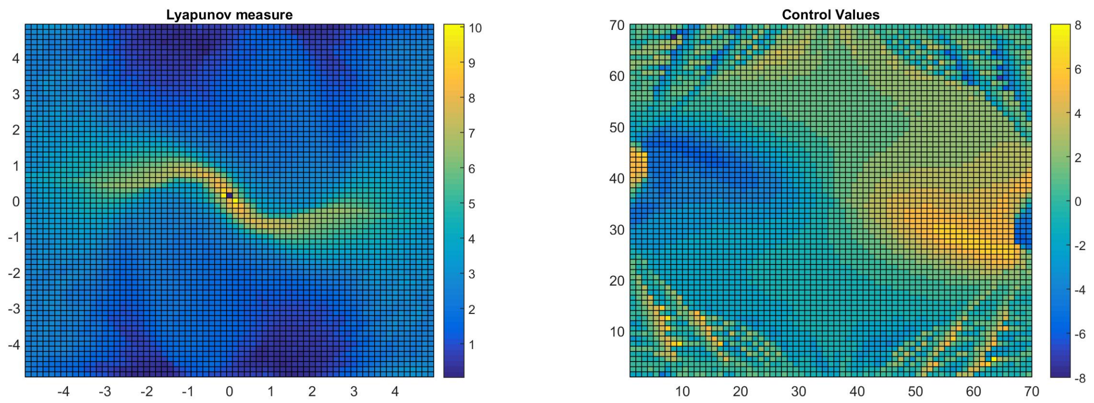
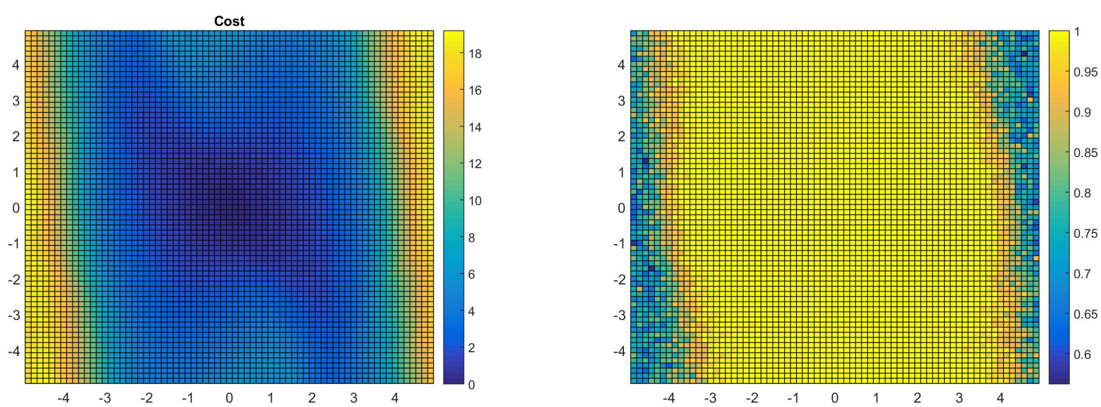
Case 2: $\sigma = 1.5$

Now, the Lyapunov measure plot is largely deviated from that of Case 1. The limit cycle is no longer distinguishable.



(a) (b)

Figure 5.11: Case 2: a) Lyapunov measure; b) Control measure



(a) (b)

Figure 5.12: Case 2: a) Cost of stabilization; b) Fraction of points stabilized

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From Fig. 5.12b, we observe few completely unstable cells in the right and left boundaries of the phase space. This is not approximation error rather the control values from $\mathcal{U}$, were not enough to stabilize the highly stochastic system in this case.

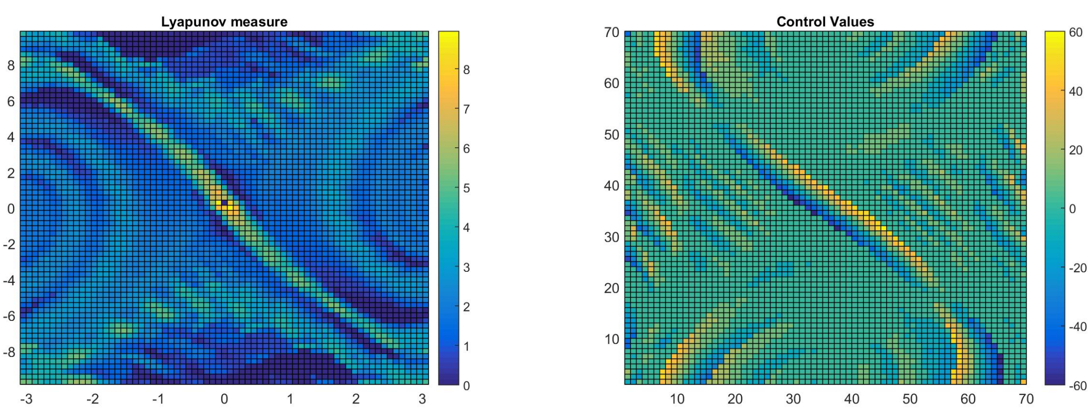
5.6.2 Uncertainty in availability of control

In this section, we study the effects of Bernoulli distribution of control values on stabilization of nonlinear systems. The probability that control value is available for the system is p means binary erasure probability is $q = 1 - p$. That implies the probability that control value was not available during stabilization is q. We took different erasure probabilities for studying the effects of Bernoulli distributed control design.

5.6.2.1 Inverted Pendulum on a Cart

The parameter b multiplying the control input is assumed to be Bernoulli random variable with statistics $Prob(b = 1) = p$ and $Prob(b = 0) = 1 - p$ for every time.

No erasure: $q = 0$



(a) (b)

Figure 5.13: a) Lyapunov measure for $q = 0$; b) Control measure for $q = 0$

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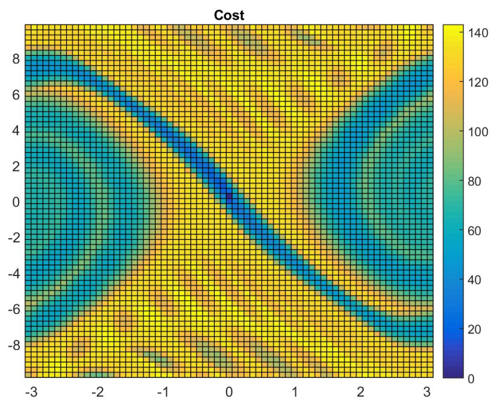
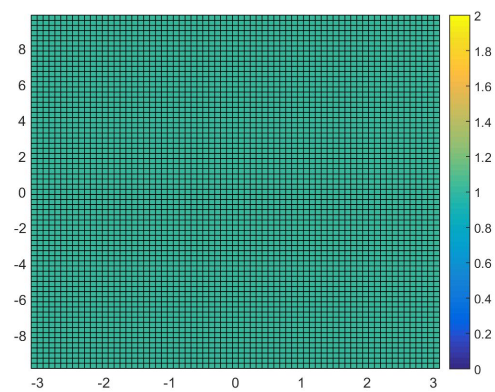
In this case, $p = 1$, which indicates control values are always available. Lyapunov measure plot 5.13a fairly resembles the phase portrait of the system. The stable manifold is visible as well.



(a)



(b)

Figure 5.14: a) Cost of stabilization for $q = 0$; b) Fraction of points stabilized for $q = 0$

From the simulation we see that, system is completely stabilized when there is no erasure of control.

Partial erasure: $q = 0.15$

In this case $p = 0.85$, so control values are available in 85% of the time. Comment: system is almost stabilized.

Lyapunov measure plot 5.15a fairly resembles the phase portrait of the system but, now it is less prominent than the previous case of zero erasure of control. The stable manifold is still visible.

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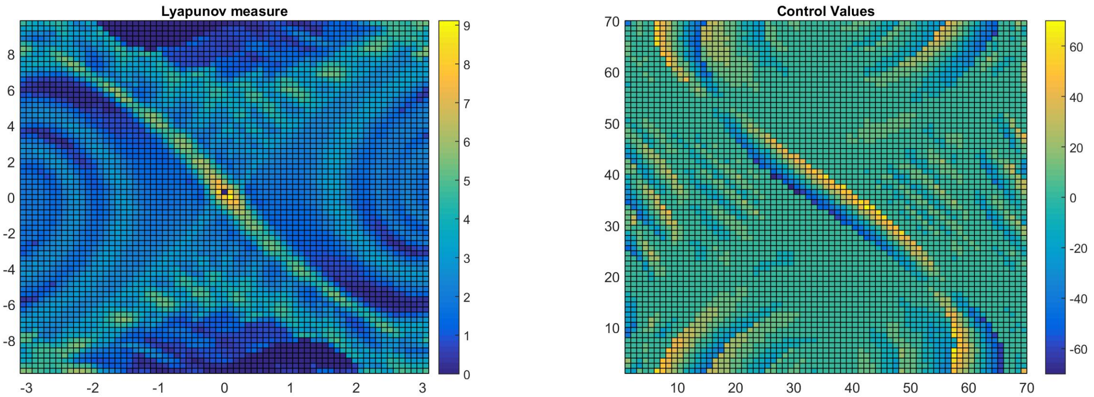
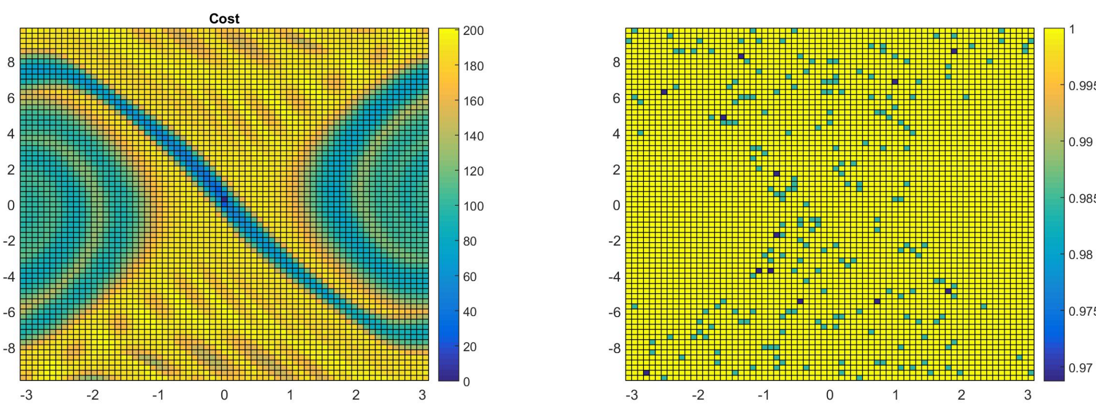


(a) (b)

Figure 5.15: a) Lyapunov measure for $q = 0.15$; b) Control measure for $q = 0.15$



(a) (b)

Figure 5.16: a) Cost of stabilization for $q = 0.15$; b) Percentage of points stabilized for $q = 0.15$

From the simulation we see that partial erasure of 15% of control values still makes the system almost stabilized. Beyond this erasure value the system starts moving more to instability.

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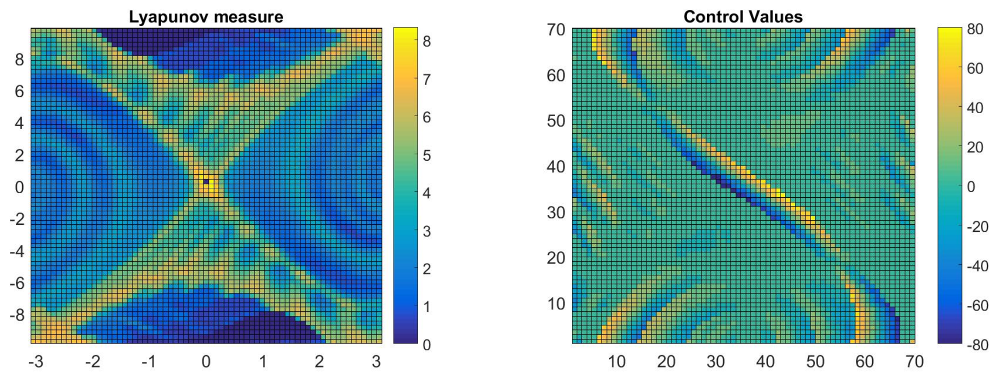
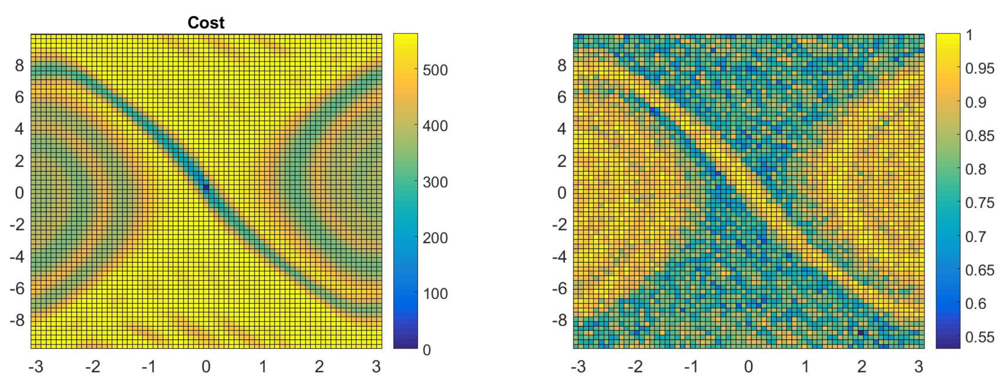
Half erasure: $q = 0.5$

In this case $p = 0.5$, so control values are available in half of the instances. System could not be stabilized. Lyapunov measure plot 5.17a largely deviated from the phase portrait of the system. Stable or unstable manifolds are no longer visible.



(a) (b)

Figure 5.17: a) Lyapunov measure for $q = 0.5$; b) Control measure for $q = 0.5$



(a) (b)

Figure 5.18: a) Cost of stabilization for $q = 0.5$; b) Percentage of points stabilized for $q = 0.5$

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From the simulation results, we see that for high erasure (50% as shown above) of control values the system can not be stabilized anymore.

However, we can observe few time domain simulations to understand how fast the trajectories are being moved towards the invariant set. For no erasure or small erasure probability of control, most of the trajectories move to the invariant set. For high erasure case ($q = 0.5$), most of the trajectories move to invariant set between 10 to 15 iterations but the overall structure of the system trajectories form a highly spanned right skewed distribution of iterations. So, there are many trajectories tending not to move to invariant set for high erasure.

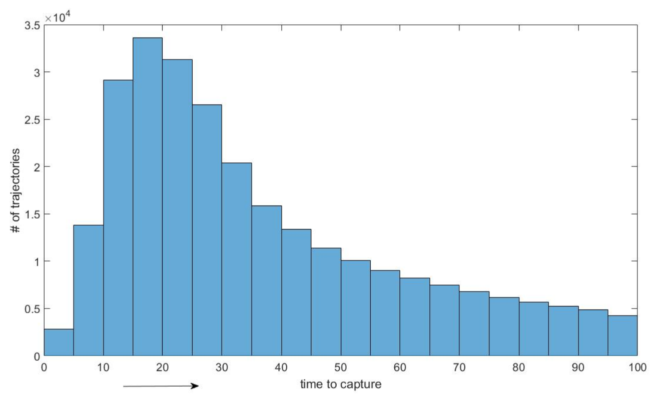

Figure 5.19: Histogram of steps to stabilize trajectories for $q = 0.5$ with 20 bins

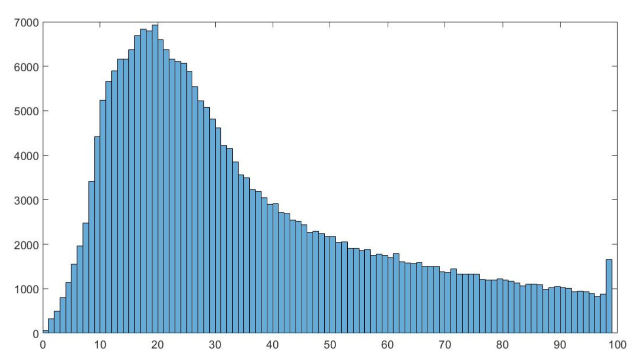

Figure 5.20: Histogram of steps to stabilize trajectories for $q = 0.5$ with 100 bins

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We plot the steps to capture the trajectories in histogram in Fig 5.19 and 5.20. Comparing these two histograms we observe that 20 equal bins captures the dynamics of trajectories quite well. For these histograms, we used 64 initial points in each of the boxes while performing the time domain simulation. That is why we could get a lot more detail on the trajectories in the discrete phase space.

5.7 Conclusions

In this paper, Transfer Perron-Frobenius operator-based framework is introduced for optimal stabilization of stochastic nonlinear systems. Weaker set-theoretic notion of almost everywhere stability is used for the design of optimal stabilizing feedback controller. The optimal stabilization problem is formulated as an infinite-dimensional linear program. The finite dimensional approximation of the linear program and the associated optimal feedback controller is obtained using set-oriented numerics. We outlined the shifts of lyapunov measure, control measure, cost and percent of stabiization using set-oriented numerical algorithm which shows with increasing stochasticity of parameters, system requires more rapid control actions for successful stabilization.

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CHAPTER 6. CONCLUSION AND FUTURE WORK

6.1 Conclusion

As we approached system stabilization, we chose to study both model based and data-driven analysis. This has potential to be useful for multitude of research problems in academia and industry since the transfer operator based stabilization has the advantage of linear approximated models. To gather the knowledge of the systems we generate Koopman operators using EDMD and NSDMD. Since Koopman operator and Perron-Frobenius operators are dual to each other, we can easily obtain the transfer Perron-Frobenius operator for the system which essentially provides the evolution of densities. This makes the stability analysis, controller design and optimal stabilization much simpler. We use Lyapunov measure based stability design which requires solving a linear program. Linear programs are easier to solve compared to other nonlinear optimization techniques which are the cases if we were not using Transfer Operator approach. Therefore, the approach described in this thesis for stability analysis and optimal stabilization makes the data-driven stability analysis a much simpler than most other popular nonlinear techniques. The choice of appropriate basis function still remains a open area for research. This approach can be extended to solve real world complex nonlinear systems in the coming days.

6.2 Discussion and Future Work

With the introduction of the data-driven stability analysis, controller design and optimal stabilization, we expect this work will be able to show paths to multiple directions of research for data-driven stabilization. Radial basis function has positivity property which was an excellent first choice for the analysis presented here. We can use other popular basis functions like Thin plate spline, different Gaussian basis functions, positivity preserving monomial basis functions etc. Per-

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forming similar analysis with any of these, one could discover the most suitable basis to choose for particular nonlinear system at hand. Discretization of state space and discretization of control are assumed in set oriented approach. Level of discretization, whether fine or coarse often depends on the system in consideration. Choosing appropriate discretization on control still concluded to be case by case basis. However, it is possible to gather intuition how coarse control should be. It would be an excellent research project to study these with more computational power using state of the art high performance computing machines and gather more knowledge of this structure which has potential to offer breakthroughs in controls research domain.

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